



# Graded Brauer tree algebras

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## ABSTRACT

In this paper we construct non-negative gradings on a basic Brauer tree algebra  $A_\Gamma$  corresponding to an arbitrary Brauer tree  $\Gamma$  of type  $(m, e)$ . We do this by transferring gradings via derived equivalence from a basic Brauer tree algebra  $A_S$ , whose tree is a star with the exceptional vertex in the middle, to  $A_\Gamma$ . The grading on  $A_S$  comes from the tight grading given by the radical filtration. To transfer gradings via derived equivalence we use tilting complexes constructed by taking Green's walk around  $\Gamma$  (cf. Schaps and Zakay-Ilouz (2001) [17]). By computing endomorphism rings of these tilting complexes we get graded algebras.

We also compute  $\text{Out}^K(A_\Gamma)$ , the group of outer automorphisms that fix the isomorphism classes of simple  $A_\Gamma$ -modules, where  $\Gamma$  is an arbitrary Brauer tree, and we prove that there is unique grading on  $A_\Gamma$  up to graded Morita equivalence and rescaling.

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## 1. Introduction

In this paper we classify gradings on Brauer tree algebras. Gradings can be transferred via derived equivalences ([15], Theorem 6.3). We show how this idea is used on the class of Brauer tree algebras. For an arbitrary Brauer tree  $\Gamma$  of type  $(m, e)$ , i.e. for a Brauer tree with  $e$  edges and multiplicity  $m$  of the exceptional vertex, we transfer the tight grading from the basic Brauer tree algebra  $A_S$  corresponding to the Brauer star  $S$  of the same type  $(m, e)$ , to the Brauer tree algebra  $A_\Gamma$ . In Section 4 we prove that the resulting grading on  $A_\Gamma$  is non-negative and we investigate its properties. In particular, this construction associates with the Brauer tree algebra  $A_\Gamma$ , which is a symmetric algebra, a quasi-hereditary algebra  $A_0$ , the subalgebra of  $A_\Gamma$  consisting of elements of  $A_\Gamma$  which have degree 0. We prove in Section 5 that the knowledge of  $A_0$  and of the cyclic ordering of its components is sufficient to recover the whole algebra  $A_\Gamma$ . In Sections 6 and 7 we give explicit formulae for the graded Cartan matrix and the graded Cartan determinant of  $A_\Gamma$ , and we prove that the graded Cartan determinant only depends on the type of the Brauer tree. Sections 9 and 10 deal with the problem of shifting the summands of a tilting complex and with the change of the exceptional vertex when the multiplicity of the exceptional vertex is 1. In the last section we compute  $\text{Out}^K(A_\Gamma)$ , the group of outer automorphisms that fix isomorphism classes of simple  $A_\Gamma$ -modules, and we prove that it only depends on the multiplicity of the exceptional vertex. We also classify all gradings on an arbitrary Brauer tree algebra, and we prove that there is unique grading up to graded Morita equivalence and rescaling.

## 2. Notation

Throughout this text,  $k$  will be an algebraically closed field of positive characteristic. All algebras will be finite dimensional algebras over  $k$  and all modules will be left modules. The category of finite dimensional  $A$ -modules will be denoted by  $A\text{-mod}$  and the full subcategory of finite dimensional projective  $A$ -modules will be denoted by  $P_A$ . The derived category of bounded complexes over  $A\text{-mod}$  will be denoted by  $D^b(A)$  and the homotopy category of bounded complexes over  $P_A$  will be denoted by  $K^b(P_A)$ .

Let  $A$  be a  $k$ -algebra. We say that  $A$  is a graded algebra if  $A$  is the direct sum of subspaces  $A = \bigoplus_{i \in \mathbb{Z}} A_i$ , such that  $A_i A_j \subset A_{i+j}$ ,  $i, j \in \mathbb{Z}$ . If  $A_i = 0$  for  $i < 0$ , we say that  $A$  is non-negatively graded. An  $A$ -module  $M$  is graded if it is the direct sum of subspaces

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$M = \bigoplus_{i \in \mathbb{Z}} M_i$ , such that  $A_i M_j \subset M_{i+j}$ , for all  $i, j \in \mathbb{Z}$ . If  $M$  is a graded  $A$ -module, then  $N = M\langle i \rangle$  denotes the graded module given by  $N_j = M_{i+j}$ ,  $j \in \mathbb{Z}$ . An  $A$ -module homomorphism  $f$  between two graded modules  $M$  and  $N$  is a homomorphism of graded modules if  $f(M_i) \subseteq N_i$ , for all  $i \in \mathbb{Z}$ . For a graded algebra  $A$ , we denote by  $A\text{-modgr}$  the category of graded finite dimensional  $A$ -modules. We set  $\text{Homgr}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{A\text{-gr}}(M, N\langle i \rangle)$ , where  $\text{Hom}_{A\text{-gr}}(M, N\langle i \rangle)$  denotes the space of all graded homomorphisms between  $M$  and  $N\langle i \rangle$  (the space of homogeneous morphisms of degree  $i$ ). There is an isomorphism of vector spaces  $\text{Hom}_A(M, N) \cong \text{Homgr}_A(M, N)$  that gives us a grading on  $\text{Hom}_A(M, N)$  (cf. [13], Corollary 2.4.4.).

Let  $X = (X^i, d^i)$  be a complex of  $A$ -modules. We say that  $X$  is a complex of graded  $A$ -modules, or just a graded complex, if for each  $i \in \mathbb{Z}$ ,  $X^i$  is a graded module and  $d^i$  is a homomorphism between graded  $A$ -modules. If  $X$  is a graded complex, then  $X\langle j \rangle$  denotes the complex of graded  $A$ -modules given by  $(X\langle j \rangle)^i := X^i\langle j \rangle$  and  $d_{X\langle j \rangle}^i := d^i$ . Let  $X$  and  $Y$  be graded complexes. A homomorphism  $f = \{f^i\}_{i \in \mathbb{Z}}$  between complexes  $X$  and  $Y$  is a homomorphism of graded complexes if for each  $i \in \mathbb{Z}$ ,  $f^i$  is a homomorphism of graded modules. The category of complexes of graded  $A$ -modules will be denoted by  $C_{\text{gr}}(A)$ . We set  $\text{Homgr}_A(X, Y) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{C_{\text{gr}}(A)}(X, Y\langle i \rangle)$ , where  $\text{Hom}_{C_{\text{gr}}(A)}(X, Y\langle i \rangle)$  denotes the space of graded homomorphisms between  $X$  and  $Y\langle i \rangle$  (the space of homogeneous morphisms of degree  $i$ ). As for modules, we have an isomorphism of vector spaces  $\text{Homgr}_A(X, Y) \cong \text{Hom}_A(X, Y)$  that gives us a grading on  $\text{Hom}_A(X, Y)$ . From this we get a grading on  $\text{Hom}_{K^b(A\text{-mod})}(X, Y)$ , since the subspace of zero homotopic maps is homogeneous. We denote this graded space by  $\text{Homgr}_{K^b(A\text{-mod})}(X, Y)$ .

Unless otherwise stated, for a graded algebra  $A$ , we will assume that the projective indecomposable  $A$ -modules are graded in such way that their tops are in degree 0. For an indecomposable bounded graded complex of projective  $A$ -modules whose leftmost non-zero term is indecomposable, we will assume that the leftmost non-zero term is graded in such way that its top is in degree 0.

We say that two symmetric algebras  $A$  and  $B$  are derived equivalent if their derived categories of bounded complexes are equivalent. From Rickard's theory we know that  $A$  and  $B$  are derived equivalent if and only if there exists a tilting complex  $T$  of projective  $A$ -modules such that  $\text{End}_{K^b(P_A)}(T) \cong B^{\text{op}}$ . For more details on derived categories and derived equivalences we recommend [11].

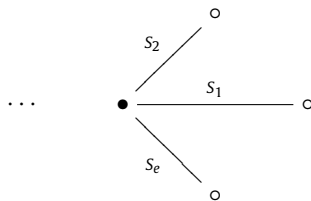
### 3. Brauer tree algebras

As a good reference for Brauer tree algebras we recommend [1].

Let  $\Gamma$  be a Brauer tree of type  $(m, e)$ , that is, a tree with  $e$  edges and multiplicity  $m$  of the exceptional vertex. If  $A$  is a Brauer tree algebra associated with  $\Gamma$ , we say that  $A$  has type  $(m, e)$ .

We will usually label the edges of a Brauer tree by the corresponding simple modules of a Brauer tree algebra associated with this tree. We will always assume that the cyclic ordering of the edges adjacent to a given vertex is given by the counter-clockwise direction in the given planar embedding of the tree.

**Example 3.1.** A very important Brauer tree of type  $(m, e)$  is the Brauer star, the Brauer tree with  $e$  edges adjacent to the exceptional vertex which has multiplicity  $m$ .



The composition factors, starting from the top, of the projective indecomposable module  $P_j$  corresponding to the edge  $S_j$  are

$$S_j, S_{j+1}, \dots, S_e, S_1, S_2, \dots, S_{j-1}, \dots, S_j, S_{j+1}, \dots, S_e, S_1, S_2, \dots, S_{j-1}, S_j$$

where  $S_j$  appears  $m + 1$  times and  $S_i$  appears  $m$  times, for  $i \neq j$ .

Any two Brauer tree algebras associated with the same Brauer tree  $\Gamma$  and defined over the same field  $k$  are Morita equivalent (cf. [11], Corollary 4.3.3).

A basic Brauer tree algebra corresponding to a Brauer tree  $\Gamma$  is isomorphic to the algebra  $kQ/I$ , where  $Q$  is a quiver and  $I$  is the ideal of relations. To construct  $Q$  and  $I$  we do the following (cf. [8], Section 5): We replace each edge of the Brauer tree by a vertex (we use the same labelling for the vertices of the quiver as for the corresponding edges of the Brauer tree) and for two adjacent edges, say  $j_1$  and  $j_2$ , which come one after the other in the circular ordering, say  $j_2$  comes after  $j_1$ , we have an arrow connecting the two corresponding vertices, starting at the vertex corresponding to  $j_2$  and ending at the vertex corresponding to  $j_1$ . If there is only one edge adjacent to the exceptional vertex and  $m > 1$ , then we add a loop starting and ending at the vertex corresponding to the edge that is adjacent to the exceptional vertex of  $\Gamma$ . This will give us the quiver  $Q$ .

Notice that, for each vertex of  $\Gamma$  that has more than one edge adjacent to it, we get a cycle in the quiver  $Q$ . The cycle of  $Q$  corresponding to the exceptional vertex will be called the exceptional cycle. If we assume that  $\Gamma$  is not the star, then the ideal  $I$  is generated by two types of relations. The relations of the first type are given by  $ab = 0$ , where arrows  $a$  and  $b$  belong

to different cycles of  $Q$ . The second type relations are the relations of the form  $\alpha = \beta$ , for two cycles  $\alpha$  and  $\beta$  starting and ending at the same vertex, neither of which is the exceptional cycle, and relations of the form  $\alpha^m = \beta$ , if  $\alpha$  is the exceptional cycle. The basic algebra  $kQ/I$  constructed in this way will be denoted by  $A_\Gamma$ .

If  $\Gamma$  is the star, then the corresponding quiver has only one cycle and the ideal of relations is generated by all paths of length  $me + 1$ . This basic algebra corresponding to the Brauer star will be denoted by  $A_S$ .

#### 4. Transfer of gradings

Let  $A$  and  $B$  be two symmetric algebras over a field  $k$  and let us assume that  $A$  is a graded algebra. The following theorem is due to Rouquier.

**Theorem 4.1** ([15], Theorem 6.3). *Let  $A$  and  $B$  be as above. Let  $T$  be a tilting complex of  $A$ -modules that induces derived equivalence between  $A$  and  $B$ . Then there exists a grading on  $B$  and a structure of a graded complex  $T'$  on  $T$ , such that  $T'$  induces an equivalence between the derived categories of graded  $A$ -modules and graded  $B$ -modules.*

This theorem tells us that derived equivalences are compatible with gradings, that is, gradings can be transferred between symmetric algebras via derived equivalences.

We will now explain how the transfer of gradings via derived equivalences is done in our context of Brauer tree algebras. Brauer tree algebras are determined up to derived equivalence by the multiplicity of the exceptional vertex and the number of edges of the tree ([14], Theorem 4.2). The basic Brauer tree algebra  $A_S$ , which corresponds to the Brauer star of type  $(m, e)$ , is naturally graded by putting all arrows in degree 1, i.e.  $A_S$  is tightly graded (see [5], Section 4, Proposition 4.4). This means that  $A_S$  is isomorphic to the graded algebra associated with the radical filtration, i.e.

$$A_S \cong \bigoplus_{i=0}^{\infty} (\text{rad } A_S)^i / (\text{rad } A_S)^{i+1}.$$

With respect to this grading the projective indecomposable  $A_S$ -modules are graded in such way that their  $i$ th radical layer is in degree  $i - 1$ .

We will transfer this grading from  $A_S$  to the basic Brauer tree algebra  $A_\Gamma$  corresponding to an arbitrary Brauer tree  $\Gamma$  of type  $(m, e)$ . We will see in the last section that all gradings on  $A_\Gamma$  can be obtained from this resulting grading via rescaling and shifting. In order to transfer the tight grading from  $A_S$  we will construct a tilting complex of  $A_S$ -modules which tilts from  $A_S$  to  $A_\Gamma$ . For a given tilting complex  $T$  of  $A_S$ -modules, which is a bounded complex of finitely generated projective  $A_S$ -modules, there exists a structure of a complex of graded  $A_S$ -modules  $T'$  on  $T$ . If  $T$  is a tilting complex that tilts from  $A_S$  to  $A_\Gamma$ , then  $\text{End}_{k^b(A_S)}(T) \cong A_\Gamma^{\text{op}}$ . Viewing  $T$  as a graded complex  $T'$ , and by computing its endomorphism ring as a graded object, we get a graded algebra which is isomorphic to the opposite algebra of the basic Brauer tree algebra  $A_\Gamma$  corresponding to  $\Gamma$ . We notice here that the choice of a grading on  $T'$  is unique up to shifting the grading of each indecomposable summand of  $T'$ . This follows from the fact that if we have two different gradings on an indecomposable module (bounded complex), then they differ only by a shift (cf. [3], Lemma 2.5.3).

##### 4.1. The tilting complex given by Green's walk

In [17], the authors give a combinatorial construction of a tilting complex that tilts from the basic Brauer tree algebra  $A_S$  corresponding to the star of type  $(m, e)$ , to the basic Brauer tree algebra  $A_\Gamma$  corresponding to an arbitrary Brauer tree of type  $(m, e)$ . The tilting complexes considered in [17] are direct sums of indecomposable complexes which have no more than two non-zero terms, and a complete classification for all such tilting complexes is given in [17].

We will use only one special tilting complex that arises in this way. It is the complex constructed by taking Green's walk (cf. [7]) around  $\Gamma$ . We construct it as follows:

Starting from the exceptional vertex of a Brauer tree  $\Gamma$  of type  $(m, e)$ , we take Green's walk around  $\Gamma$  and enumerate all the edges of  $\Gamma$ . We start this enumeration from an arbitrary edge adjacent to the exceptional vertex and walk around  $\Gamma$  in the counter-clockwise direction. We will eventually show that the resulting grading on  $A_\Gamma$  does not depend on where we start the enumeration. Define  $T$ , a tilting complex of  $A_S$ -modules, to be the direct sum of the complexes  $T_i$ ,  $1 \leq i \leq e$ , which correspond to the edges of  $\Gamma$ , and which are defined by induction on the distance of an edge from the exceptional vertex in the following way:

(a) If  $i$  is an edge which is adjacent to the exceptional vertex, then  $T_i$  is defined to be the stalk complex

$$Q_i : 0 \longrightarrow P_i \longrightarrow 0$$

with  $P_i$  in degree 0;

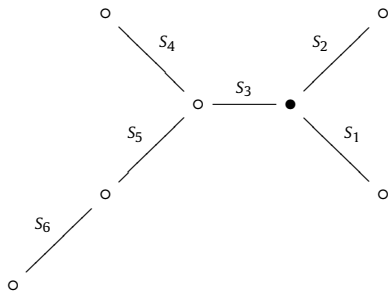
(b) If  $i$  is not adjacent to the exceptional vertex and assuming that the shortest path connecting  $i$  to the exceptional vertex is  $j_1, j_2, \dots, j_t, i$ , where  $j_1 < j_2 < \dots < j_t < i$  in the labelling from Green's walk, then  $T_i$  is defined to be the complex  $Q_{j_t i}[n_i]$ , where  $Q_{j_t i}$  is the following complex with two non-zero entries in degrees 0 and 1

$$Q_{j_t i} : 0 \longrightarrow P_{j_t} \xrightarrow{h_{j_t i}} P_i \longrightarrow 0,$$

where  $h_{j_t i}$  is a homomorphism of the highest possible rank, and  $n_i$  is the shift necessary to ensure that  $P_{j_t}$  is in the same degree as  $P_{j_t}$  in other summands of  $T$  which are previously determined.

For the convenience of the reader we include the following example.

**Example 4.2.** Let  $\Gamma$  be the following Brauer tree with multiplicity 1 and with edges numbered by taking Green's walk:



The tilting complex of  $A_S$ -modules, where  $S$  is the Brauer star with six edges and multiplicity 1, given by taking Green's walk around  $\Gamma$  is the direct sum  $T = \bigoplus_{i=1}^6 T_i$ , where  $T_1, T_2$  and  $T_3$  are the stalk complexes  $P_1, P_2$  and  $P_3$  respectively, in degree 0,  $T_4$  is  $P_3 \xrightarrow{h_{34}} P_4$  with  $P_3$  in degree 0,  $T_5$  is  $P_3 \xrightarrow{h_{35}} P_5$  with  $P_3$  in degree 0, and  $T_6$  is  $P_5 \xrightarrow{h_{56}} P_6$  with  $P_5$  in degree 1. This complex tilts from  $A_S$  to  $A_\Gamma$ , i.e.  $\text{End}_{K^b(P_{A_S})}(T) \cong A_\Gamma^{\text{op}}$ .

#### 4.2. Calculating $\text{End}_{K^b(P_{A_S})}(T)$

Let  $A_S$  be a basic Brauer tree algebra corresponding to the star of type  $(m, e)$  and let  $A_\Gamma$  be a basic Brauer tree algebra corresponding to a given Brauer tree  $\Gamma$  of type  $(m, e)$ . Let  $T$  be the tilting complex of  $A_S$ -modules that tilts from  $A_S$  to  $A_\Gamma$ , constructed as in the previous section, i.e. constructed by taking Green's walk around  $\Gamma$ . Viewing each summand  $T_i$  of  $T$  as a graded complex  $T'_i$ , we have a structure of a graded complex  $T'$  on  $T$ . By calculating  $\text{Endgr}_{K^b(P_{A_S})}(T') \cong A_\Gamma^{\text{op}}$  we get  $A_\Gamma$  as a graded algebra. We will choose  $T'_i$  to be  $T_i(r_i)$ , where  $r_i$  will be the necessary shifts that will ensure that the resulting grading is non-negative. We remind the reader that  $T_i$  is assumed to be graded in such way that its leftmost non-zero term has its top in degree 0.

We now state the main theorem of this section.

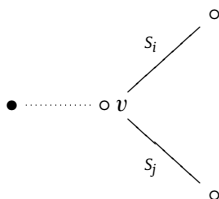
**Theorem 4.3.** Let  $\Gamma$  be an arbitrary Brauer tree with  $e$  edges and multiplicity  $m$  of the exceptional vertex and let  $A_\Gamma$  be a basic Brauer tree algebra determined by this tree. The algebra  $A_\Gamma$  can be non-negatively graded.

**Proof.** In order to grade  $A_\Gamma$ , we need to calculate  $\text{Homgr}_{K^b(P_{A_S})}(T'_i, T'_j)$ , as graded vector spaces, for those  $T'_i$  and  $T'_j$  which correspond to edges  $S_i$  and  $S_j$  that are adjacent to the same vertex of  $\Gamma$ , and which come one after the other,  $i$  after  $j$ , in the circular ordering associated with that vertex. This is a consequence of the fact that when identifying  $\text{End}_{K^b(P_{A_S})}(T)^{\text{op}}$  with  $A_\Gamma$ , elements corresponding to the vertices of the quiver of  $A_\Gamma$  are given by  $\text{id}_{T_i} \in \text{End}_{K^b(P_{A_S})}(T_i)$ ,  $i = 1, 2, \dots, e$ ; and the subspace of  $A_\Gamma$  generated by all paths starting at the vertex of  $A_\Gamma$  corresponding to  $\text{id}_{T_i} \in \text{End}_{K^b(P_{A_S})}(T_i)$  and finishing at the vertex corresponding to  $\text{id}_{T_j} \in \text{End}_{K^b(P_{A_S})}(T_j)$ , is given by  $\text{Hom}_{K^b(P_{A_S})}(T_i, T_j)$ . In fact, we only need to calculate the non-zero summand of  $\text{Homgr}_{K^b(P_{A_S})}(T'_i, T'_j)$  which is in the lowest degree. That will be the degree of the unique arrow of the quiver of  $A_\Gamma$  pointing from the vertex corresponding to  $S_i$  to the vertex corresponding to  $S_j$ .

The dimension of  $\text{Hom}_{K^b(P_{A_S})}(T_i, T_j)$

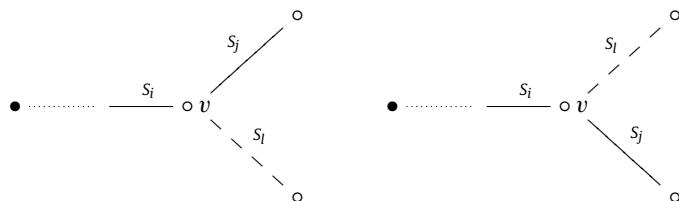
- (1) is 0, if the vertices corresponding to  $\text{id}_{T_i}$  and  $\text{id}_{T_j}$  do not belong to the same cycle,
- (2) (2.a) is  $m$ , if  $i \neq j$  and the vertices corresponding to  $\text{id}_{T_i}$  and  $\text{id}_{T_j}$  belong to the exceptional cycle,  
 (2.b) is 1, if  $i \neq j$  and the vertices corresponding to  $\text{id}_{T_i}$  and  $\text{id}_{T_j}$  belong to the same non-exceptional cycle,
- (3) (3.a) is  $m + 1$ , if  $i = j$  and the vertex corresponding to  $\text{id}_{T_i}$  belongs to the exceptional cycle,  
 (3.b) is 2, if  $i = j$  and the vertex corresponding to  $\text{id}_{T_i}$  does not belong to the exceptional cycle.

Edges  $i$  and  $j$  of a Brauer tree  $\Gamma$  that are adjacent to the same vertex, say  $v$ , and that come one after the other in the circular ordering of  $v$ , can either be at the same distance from the exceptional vertex or the distance of one of them, say  $i$ , is one less than the distance of  $j$ . If the former holds, then a part of  $\Gamma$  is given by



where  $v$  may or may not be the exceptional vertex.

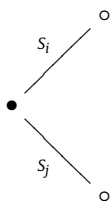
If the latter holds, then a part of  $\Gamma$  is given by one of the following two diagrams



where the leftmost vertex of  $S_i$  may or may not be the exceptional vertex, and there may or may not be more edges, such as  $S_l$ , adjacent to  $v$ . In the case of the first diagram we have an arrow from  $S_i$  to  $S_j$  in the quiver of  $A_\Gamma$ , and in the case of the second diagram we have an arrow from  $S_j$  to  $S_i$ .

It follows that it is sufficient to consider the following four cases:

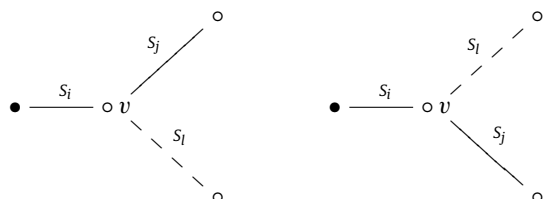
Case 1. Edges  $S_i$  and  $S_j$  are adjacent to the exceptional vertex. The corresponding part of  $\Gamma$  is



In this case, the corresponding summands of the graded tilting complex  $T'$  are  $T'_i := T_i$  and  $T'_j := T_j$ , where  $T_i$  and  $T_j$  are stalk complexes  $P_i$  and  $P_j$  concentrated in degree 0. If  $i > j$ , then  $\text{Homgr}_{K^b(P_A)}(T'_i, T'_j) \cong \text{Homgr}_{A_S}(P_i, P_j) \cong k\langle -(i-j) \rangle \oplus M$ , as graded vector spaces, where  $M$  is the sum of the summands that appear in higher degrees than  $i-j$ . In other words,  $i-j$  is the degree of the corresponding arrow of the quiver of  $A_\Gamma$  whose source is  $S_i$  and whose target is  $S_j$ . If  $i < j$ , then the degree of the corresponding arrow of the quiver of  $A_\Gamma$  is  $e - (j-i)$ , where  $e$  is the number of edges of the tree.

If there is only one edge adjacent to the exceptional vertex and  $m > 1$ , then the corresponding loop will be in degree  $e$ .

Case 2. Edges  $S_i$  and  $S_j$  are adjacent to a non-exceptional vertex  $v$  and one of them, say  $S_i$ , is adjacent to the exceptional vertex. Then we have that a part of  $\Gamma$  is given by one of the following two diagrams



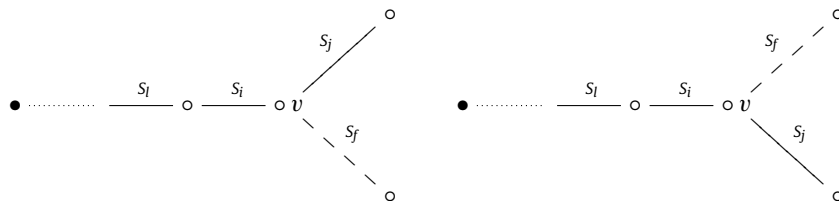
where it could happen that there are more edges adjacent to  $v$ , such as  $S_l$ .

In this case, the summands of the tilting complex  $T'$  are  $T'_i := Q_i$ , the stalk complex with  $P_i$  in degree 0, and  $T'_j := Q_{ij}$ , where  $Q_{ij}$  is previously defined complex  $P_i \xrightarrow{h_{ij}} P_j$ , with  $P_i$  in degree 0.

In case of the first diagram, there is an arrow from  $S_i$  to  $S_j$  in the quiver of  $A_\Gamma$ . Since  $i < j$ , we have that any morphism from  $Q_i$  to  $Q_{ij}$  has to map  $P_i$  to the kernel of the map  $h_{ij}$ . It follows that this map has to map  $\text{top } P_i$  to  $\text{soc } P_i$ . Since  $\text{soc } P_i$  is in degree  $me$ , it follows that the corresponding arrow of the quiver of  $A_\Gamma$  is in degree  $me$ .

In case of the second diagram, there is an arrow from  $S_j$  to  $S_i$  in the quiver of  $A_\Gamma$ . The identity map on  $P_i$  will give us a non-zero morphism between  $Q_{ij}$  and  $Q_i$ . It follows that the corresponding arrow is in degree 0.

Case 3. Edges  $S_i$  and  $S_j$ , where  $i < j$ , are adjacent to a non-exceptional vertex  $v$ ,  $S_i$  is closer to the exceptional vertex than  $S_j$ , and the leftmost vertex of  $S_i$  is non-exceptional. Then we have that a part of  $\Gamma$  is given by one of the following two diagrams



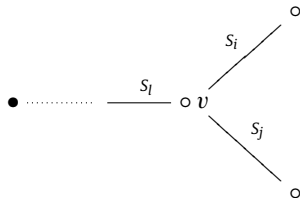
where it could happen that there are more edges adjacent to  $v$ , such as  $S_f$ . The leftmost vertex of  $S_i$  can be either exceptional or non-exceptional.

The summands  $T'_i$  and  $T'_j$  of the graded tilting complex  $T'$  corresponding to the edges  $S_i$  and  $S_j$  are defined to be the graded complexes  $(Q_{ii}[n_i])\langle r_i \rangle$  and  $(Q_{jj}[n_j])\langle r_j \rangle$ , where the vertical shift  $\langle r_i \rangle$  (respectively  $\langle r_j \rangle$ ) is necessary to ensure that the top of the leftmost term of  $Q_{ii}$  (respectively  $Q_{jj}$ ), which is  $P_i$  (respectively  $P_j$ ), is in the same degree as the top of  $P_i$  (respectively  $P_j$ ) in other summands of  $T$  which have been previously defined. We remind the reader that  $Q_{ii}[n_i]$  has been previously defined at the beginning of this section. This way we avoid negative degrees.

If the first diagram occurs, then any morphism of graded complexes from  $(Q_{ii}[n_i])\langle r_i \rangle$  to  $(Q_{jj}[n_j])\langle r_j \rangle$  has to map top  $P_i$  to soc  $P_j$ . From this we conclude that the corresponding arrow of the quiver of  $A_\Gamma$  that points from  $S_i$  to  $S_j$  is in degree  $me$ .

If the second diagram occurs, then the identity map on  $P_i$  gives us a morphism from  $(Q_{jj}[n_j])\langle r_j \rangle$  to  $(Q_{ii}[n_i])\langle r_i \rangle$  which is not homotopic to zero. From this we have that the arrow from  $S_j$  to  $S_i$  is in degree 0.

Case 4. Edges  $S_i$  and  $S_j$ , where  $j < i$ , are adjacent to a non-exceptional vertex  $v$ , and the edge with minimal index among the edges adjacent to  $v$ , say  $S_i$ , comes before  $S_j$  in the circular ordering of  $v$ . Then we have that a part of  $\Gamma$  is

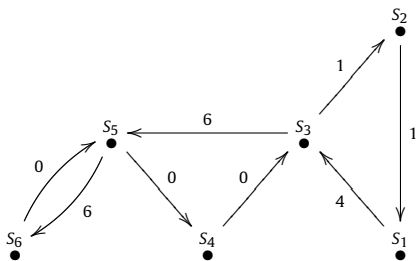


where the leftmost vertex of  $S_i$  can be either exceptional or non-exceptional. In each case, the summands  $T'_i$  and  $T'_j$  of the graded tilting complex  $T'$  corresponding to the edges  $S_i$  and  $S_j$  are defined to be  $(Q_{ii}[n_i])\langle r_i \rangle$  and  $(Q_{jj}[n_j])\langle r_j \rangle$ , where  $r_i$  is defined as in the previous case. The map  $(\text{id}_{P_i}; h_{ij})$ , where  $h_{ij}$  is a map of the maximal rank, from  $(Q_{ii}[n_i])\langle r_i \rangle$  to  $(Q_{jj}[n_j])\langle r_j \rangle$  will give us a non-zero morphism in  $\text{Homgr}_{K^b(P_{A_\Gamma})}(T'_i, T'_j)$ . This map is in degree 0 because  $\text{id}_{P_i}$  is a homogeneous map of degree zero. Therefore, the corresponding arrow from  $S_i$  to  $S_j$  is in degree 0.

These four cases cover all possible local structures of a Brauer tree  $\Gamma$  that we can encounter when putting a grading on the basic Brauer tree algebra  $A_\Gamma$  that corresponds to this tree. We only need to walk around the Brauer tree  $\Gamma$  and recognize which of the four cases occurs for the adjacent edges  $S_i$  and  $S_j$ . In each of the four cases above, we have that the corresponding arrows are in non-negative degrees. Hence, the resulting grading on  $A_\Gamma$  is non-negative. ■

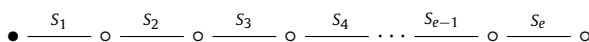
The grading on  $A_\Gamma$  constructed in the proof of the previous theorem will be referred to as the grading constructed by taking Green's walk around  $\Gamma$ .

**Example 4.4.** (a) Let  $\Gamma$  be the Brauer tree from Example 4.2. We first construct the quiver of the basic Brauer tree algebra  $A_\Gamma$  corresponding to this tree. Each edge is replaced by a vertex and for two adjacent edges, which come one after the other in the circular ordering, we have an arrow connecting two corresponding vertices of the quiver in the opposite ordering of the circular ordering of edges, i.e. in the clockwise direction.

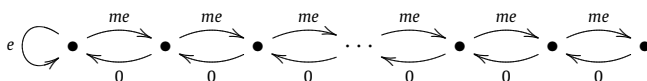


The degrees of the arrows between  $S_1$  and  $S_3$ ,  $S_3$  and  $S_2$ ,  $S_2$  and  $S_1$  are computed using the first case from the proof of the previous theorem. The degree of the arrow between  $S_3$  and  $S_5$  is 6 and is computed using the second case. The arrows between  $S_5$  and  $S_4$ , and  $S_4$  and  $S_3$  are in degree 0 and these degrees are computed as in the fourth and the second case respectively. The degree of the arrow between  $S_5$  and  $S_6$  is 6 and the degree of the arrow between  $S_6$  and  $S_5$  is 0 and these are computed as in the third case.

(b) If the Brauer tree is the line with  $e$  edges and the exceptional vertex at one of the ends, i.e.



then the basic Brauer tree algebra  $A_\Gamma$  is graded and its quiver has  $e$  vertices



If  $m = 1$ , then there is no loop in the above quiver.

Let  $\Gamma$  be an arbitrary Brauer tree. Each edge of  $\Gamma$  that is adjacent to the exceptional vertex determines a connected subtree of  $\Gamma$  in the following way. Let  $E$  be an edge that is adjacent to the exceptional vertex. A connected subtree determined by  $E$  contains all edges of  $\Gamma$  which have property that the shortest path that connects them to the exceptional vertex contains  $E$ . We call these subtrees components of the Brauer tree  $\Gamma$ .

**Lemma 4.5.** *Let  $\alpha$  be an arrow of the exceptional cycle of the quiver of  $A_\Gamma$  which starts at  $S_i$  and ends at  $S_j$ . If  $A_\Gamma$  is graded by taking Green's walk around  $\Gamma$ , then the degree of  $\alpha$  is equal to the number of edges in the component of  $\Gamma$  corresponding to  $S_j$ .*

**Proof.** If  $i > j$ , then because of the way we enumerate edges by taking Green's walk we have that  $i = j + s$ , where  $s$  is the number of edges in the component corresponding to  $S_j$ . Hence,  $\alpha$  is in degree  $i - j = s$ .

If  $i < j$ , which only happens if  $i = 1$ , then  $\alpha$  is in degree  $e - (j - i) = e - j + 1$ , and this number is equal to the number of edges of the component corresponding to  $S_j$ . ■

From this lemma it follows that the resulting grading of the exceptional cycle does not depend on from which edge adjacent to the exceptional vertex we start enumeration of edges. This leads us to the following proposition.

**Proposition 4.6.** *Let  $A_\Gamma$  be a basic Brauer tree algebra whose tree is  $\Gamma$  and let us assume that  $A_\Gamma$  is graded by taking Green's walk around  $\Gamma$ . The resulting grading does not depend on from which edge adjacent to the exceptional vertex we start Green's walk.*

**Proof.** Let us assume that we have done two walks around  $\Gamma$  starting at a different edge each time. Let us assume that the index of  $S$ , where  $S$  is one of the edges adjacent to the exceptional vertex, is 1 in the first walk, and that its index is  $1 + l$  in the second walk. Let us assume that we got two tilting complexes  $T_1$  and  $T_2$  of  $A_S$ -modules by taking these two walks. These complexes are equal up to cyclic permutation of indices of edges of the Brauer star  $S$ . In other words, each index of each term (which is a projective indecomposable  $A_S$ -module) of each summand of  $T_1$  has been cyclically permuted by  $l$  to get the corresponding index of the corresponding term of the corresponding summand of  $T_2$ . These two tilting complexes will give us the same grading because of the cyclic structure of  $A_S$ . ■

The following two lemmas follow from the proof of Theorem 4.3.

**Lemma 4.7.** *Let  $A_\Gamma$  be a basic Brauer tree algebra whose tree is  $\Gamma$ . Let  $Q$  be its quiver and let us assume that  $A_\Gamma$  is graded by taking Green's walk around  $\Gamma$ . The only cycle of  $Q$  that does not contain any arrows that are in degree 0 is the exceptional cycle. For a non-exceptional cycle there is exactly one arrow that is not in degree 0. This arrow is in degree  $me$  and the index of its target is greater than the index of its source.*

**Lemma 4.8.** *Let  $A_\Gamma$  and  $Q$  be as above. Let  $\alpha$  be an arrow of a non-exceptional cycle of  $Q$  which is in degree 0. Then the index of the source of  $\alpha$  is greater than the index of the target of  $\alpha$ .*

**Lemma 4.9.** *Let  $A_\Gamma$  be graded as above. The socle of  $A_\Gamma$  is in degree  $me$ .*

**Proof.** For an arbitrary cycle, say  $\gamma$ , let  $\alpha_1, \dots, \alpha_r$  be the arrows of that cycle, appearing in that cyclic ordering. If  $\gamma$  is a non-exceptional cycle, then paths of the form  $\alpha_i \alpha_{i+1} \dots \alpha_{i+r-1}$ , where the addition in indices is modulo  $r$ , and  $1 \leq i \leq r$ , belong to the socle. If  $\gamma$  is the exceptional cycle, then the paths  $(\alpha_i \alpha_{i+1} \dots \alpha_{i+r-1})^m$  belong to the socle. These elements span the socle of  $A_\Gamma$ . For a non-exceptional cycle, the only arrow which is in a non-zero degree is the arrow whose source is the vertex of that cycle with the minimal index, and whose target is the vertex of that cycle with the maximal index. By the proof of Theorem 4.3, that arrow is in degree  $me$ . It follows that the path  $\alpha_i \alpha_{i+1} \dots \alpha_{i+r-1}$ , for all  $i$ , is in degree  $me$ . For the exceptional cycle, the sum of degrees of the arrows of the exceptional cycle is  $e$ , by Lemma 4.5. Hence,  $(\alpha_i \alpha_{i+1} \dots \alpha_{i+r-1})^m$  is in degree  $me$ . ■

## 5. The subalgebra $A_0$

Let  $\Gamma$  be a given Brauer tree and let  $A_\Gamma$  be a basic Brauer tree algebra associated with this tree. Let us assume that  $A_\Gamma$  has been graded by taking Green's walk around  $\Gamma$ . Then the subalgebra  $A_0$  of  $A_\Gamma$  consisting of the elements that are in degree 0 has an interesting structure.

The quiver of the basic algebra  $A_\Gamma$  is the union of the cycles contained in it. From Lemma 4.7 follows that the only cycle that does not contain any arrows that are in degree 0 is the exceptional cycle. If we assume that there are  $t$  edges that are adjacent to the exceptional vertex of  $\Gamma$ , we see immediately that the exceptional cycle divides the quiver of the subalgebra  $A_0$  into  $t$  disjoint parts, each labelled by a vertex of the exceptional cycle corresponding to an edge adjacent to the exceptional vertex of  $\Gamma$ .

**Proposition 5.1.** *Let  $A_\Gamma$  be a basic Brauer tree algebra associated with a given Brauer tree  $\Gamma$  and let  $A_0$  and  $t$  be as above. The algebra  $A_0$  is the direct product of  $t$  subalgebras.*

**Proof.** The factors of  $A_0$  are the quiver algebras of  $t$  disjoint subquivers of the quiver of  $A_0$ . ■

Each of these factors in the previous proposition is labelled by the corresponding vertex which belongs to the exceptional cycle. Let  $A_v$  be the connected component of  $A_0$  that corresponds to a vertex  $v$  of the exceptional cycle.



**Lemma 5.2.** *In the quiver of the component  $A_v$  of  $A_0$  there is at most one arrow with vertex  $v$  as its target. For any other vertex of the quiver of  $A_v$  there are at most two arrows whose target is that vertex.*

**Proof.** In the quiver  $Q$  of the basic Brauer tree algebra  $A_\Gamma$ , each vertex is contained in at most two cycles. Hence, for an arbitrary vertex  $w$  of  $Q$ , there are at most two arrows of  $Q$  whose target is  $w$ . By Lemma 4.5, if  $v$  belongs to the exceptional vertex, then one of those arrows is in a positive degree. ■

**Lemma 5.3.** *Let  $w$  be a vertex of the quiver of  $A_v$  different from  $v$ . In the quiver of  $A_v$  there is exactly one arrow that has  $w$  as its source.*

**Proof.** The vertex  $w$  belongs either to one or to two cycles of  $Q$ , depending on whether the corresponding edge of the Brauer tree is an end edge or not. Therefore, there are either one or two arrows that have  $w$  as its source. By Lemma 4.8 exactly one of those arrows is in degree 0. ■

**Proposition 5.4.** *The quiver of  $A_v$  is a directed rooted tree with vertex  $v$  as its root, and with arrows pointing from higher levels of the tree to lower levels of the tree, with root  $v$  being in level 0.*

**Proof.** The quiver of  $A_0$  does not have any cycles, because one arrow in each non-exceptional cycle of the quiver of  $A_\Gamma$  is in a positive degree. It follows from the previous two lemmas that this component is a tree with at most two arrows having the same target, and at most one arrow having an arbitrary vertex as its source. If we view this tree as a rooted tree with the vertex that belongs to the exceptional cycle as the root, then all arrows point from the higher levels to the lower levels by Lemma 4.8, with the root being in level 0. ■

**Proposition 5.5.** *Each of the components of the subalgebra  $A_0$  is the path algebra of a directed rooted tree with arrows pointing from higher levels towards lower levels. The only relations that occur in these components are of the form  $\alpha\beta = 0$ , where  $\alpha$  and  $\beta$  are arrows that belong to different cycles of the quiver of  $A_\Gamma$ , such that the target of  $\alpha$  is the source of  $\beta$ .*

**Proof.** It is left to prove that we only have relations of type  $\alpha\beta = 0$ . These relations are inherited from the relations of the algebra  $A_\Gamma$ . The only other relations that appear in  $A_\Gamma$  are of type  $\rho = \sigma$ , where  $\rho$  and  $\sigma$  are two cycles having the same source and the same target. Since in the quiver of  $A_0$  there are no cycles, these relations are not present. ■

In the following Corollary we record another important property of  $A_0$ . We refer the reader to [2], Section 9.6 for details on gentle algebras.

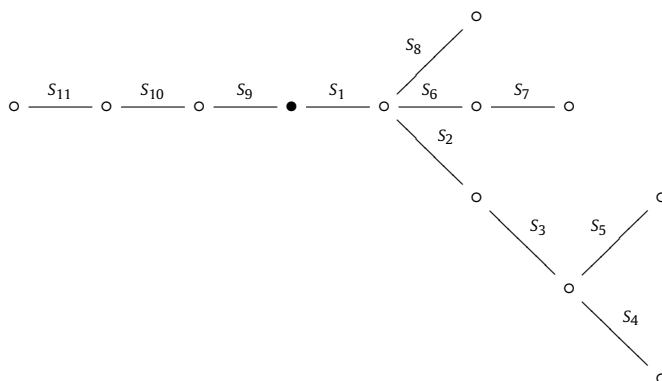
**Corollary 5.6.** *The subalgebra  $A_0$  is a gentle algebra.*

**Proof.** Follows from the previous two lemmas and two propositions, and the observation that for each arrow  $\alpha$  of the quiver of  $A_0$ , there is at most one arrow  $\beta$  and at most one arrow  $\gamma$  such that  $\beta\alpha \in I$  and  $\gamma\alpha \notin I$ , where  $I$  is the ideal of relations of  $A_0$ . ■

**Corollary 5.7.** *The subalgebra  $A_0$  is tightly graded.*

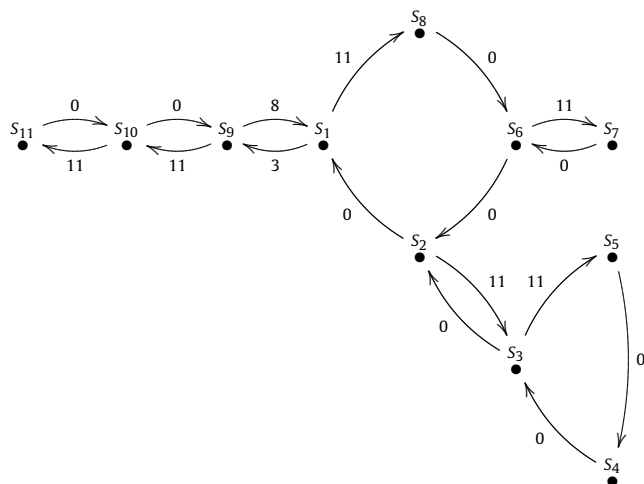
**Proof.** By the previous proposition, the ideal of relations of  $A_0$  is generated by elements of the form  $\alpha\beta$ . These elements are homogeneous regardless of the degrees of the arrows. ■

**Example 5.8.** If  $\Gamma$  is the following Brauer tree with multiplicity 1 and edges numbered by taking Green's walk:





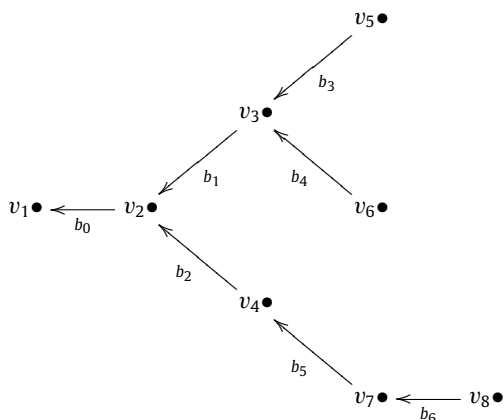
then the quiver of the graded basic Brauer tree algebra  $A_{\Gamma}$  associated with this tree is



The algebra  $A_0$  is consisted of two components because there are two edges adjacent to the exceptional vertex. The quiver of the first component is



and the only relation is  $a_1 a_2 = 0$ . The quiver of the second component is



and the relations are  $b_2 b_0 = 0$ ,  $b_5 b_2 = 0$  and  $b_4 b_1 = 0$ .

### 5.1. Recovering the quiver of $A_{\Gamma}$ from the quiver of $A_0$

The grading resulting from taking Green's walk has some interesting properties. We will see in this section that the algebra  $A_0$  carries a lot of information about  $A_{\Gamma}$ .

Let  $\Gamma$ ,  $A_{\Gamma}$  and  $A_0$  be as before. If we omit the arrows of the exceptional cycle of the quiver of  $A_{\Gamma}$ , the resulting quiver consists of the connected components which correspond to the connected components of the quiver of  $A_0$ . If we look at the components of  $A_0$  we see that it is sufficient to know the quiver and relations of such a component to recover the quiver of the corresponding component of  $A_{\Gamma}$ . This is a consequence of Lemma 4.7, which says that in every non-exceptional cycle of the quiver of  $A_{\Gamma}$  there is only one arrow that is not in degree 0.

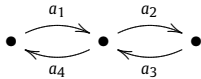
**Proposition 5.9.** *Let  $A_v$  be a connected component of  $A_0$  and let  $Q_v$  be its quiver. Let  $Q_1$  be the corresponding connected component of the quiver of  $A_{\Gamma}$  when the exceptional cycle is omitted. From the quiver and relations of  $A_v$  we can recover the quiver  $Q_1$  and the relations of  $A_{\Gamma}$  that involve only arrows of  $Q_1$ .*

**Proof.** Start from the root  $v$  of the rooted tree  $Q_v$ . Let  $\rho$  be the longest non-zero path ending at  $v$ . Add an arrow pointing from  $v$  to the source vertex of  $\rho$ . If there is no such path of length greater than 1, then add an arrow from  $v$  to the starting point of the only arrow ending at the root  $v$ . In this way we recover the cycle of  $Q_1$  which has root  $v$  as one of its vertices. The added arrow was an arrow of  $Q_1$  that is in a non-zero degree. We repeat the same step with an arbitrary vertex in level 1 of

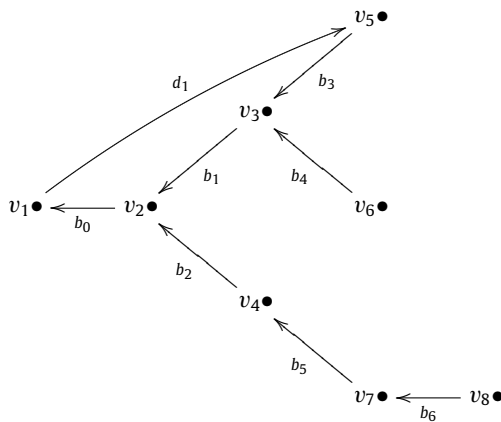
the rooted tree instead of the root, but we only consider paths which do not contain arrows that belong to already recovered cycles. Repeat the same step for all other vertices in level 1 of the rooted tree  $Q_v$ . Repeat the same steps for vertices in other levels of the rooted tree  $Q_v$ , until every cycle is recovered. In this way we recover the whole corresponding component  $Q_1$  of the quiver of  $A_\Gamma$ . As far as the relations are concerned, we get relations for the basic Brauer tree algebra corresponding to a given tree, i.e. for two successive arrows belonging to two different cycles we set their product to be zero and we set two cycles starting and ending at the same vertex to be equal. ■

**Example 5.10.** Let  $A_\Gamma$  be a basic Brauer tree algebra corresponding to the Brauer tree from Example 5.8. We have seen that  $A_0$  has two components. Let us recover the corresponding components of the quiver of  $A_\Gamma$ .

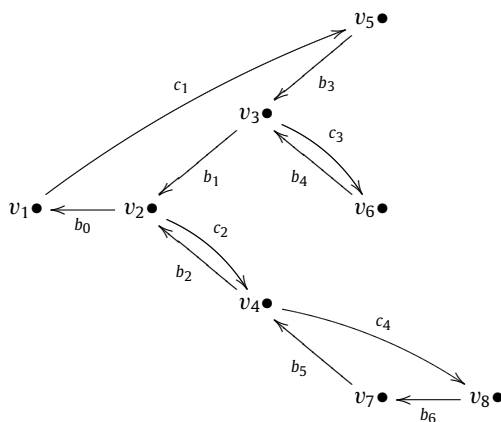
Starting from the root of the first component we immediately recover the first cycle since there is no non-zero path of length greater than 1 whose target is the root. Consequently, the second cycle is easily recovered and we get that the corresponding component of the quiver of  $A_\Gamma$  is



In the second component, the longest non-zero path ending at  $v_1$  is  $b_3b_1b_0$ . Therefore we have to add an arrow from  $v_1$  to  $v_5$ . This will give us the following partial quiver



We move on to the next level and conclude that we need to add an arrow from  $v_2$  to  $v_4$ . We do not add an arrow from  $v_2$  to  $v_5$  because the arrow from  $v_3$  to  $v_2$  is already in a fully recovered cycle. For level two vertices we need to add an arrow from  $v_3$  to  $v_6$  and an arrow from  $v_4$  to  $v_8$ . Finally, the recovered component is given by the quiver



**Theorem 5.11.** Let  $A_\Gamma$  be a graded basic Brauer tree algebra whose grading is constructed by taking Green's walk around  $\Gamma$ . From the quiver and relations of  $A_0$  and the cyclic ordering of the components of  $A_0$  we can recover the quiver and relations of  $A_\Gamma$ .

**Proof.** We have seen in the previous proposition that from the quiver and relations of  $A_0$  we can recover each of the components of the quiver of  $A_\Gamma$  that we get when we omit the exceptional cycle. In order to completely recover the quiver of  $A_\Gamma$ , we are left to recover the exceptional cycle. From the cyclic ordering of the components we get the cyclic ordering of the vertices of the exceptional cycle. Thus, the exceptional cycle is recovered. ■

## 5.2. Quasi-hereditary structure on $A_0$

Let  $Q_v$  be the quiver of an arbitrary connected component  $A_v$  of  $A_0$ . We have seen that  $Q_v$  is a rooted tree. We can enumerate the vertices of  $Q_v$  in a natural way by the levels of the rooted tree. We start with root  $v$ , then we enumerate all vertices that are in level 1 of the rooted tree, for example, we enumerate them from left to right in the planar embedding of the tree. Once we have enumerated all vertices of an arbitrary level  $r$ , we move on to level  $r + 1$  and repeat the same procedure until we enumerate all vertices. Let  $P_i$  be the projective cover of the simple  $A_0$ -module  $S_i$  corresponding to the vertex  $v_i$ . Then  $P_i$  is spanned by paths of  $Q_v$  ending at  $v_i$ . Since  $Q_v$  is a rooted tree, we conclude that the only simple modules that can occur as composition factors of  $P_i$  are the simple modules whose corresponding vertex has index greater than  $i$ . Also,  $S_i$  occurs only once as a composition factor of  $P_i$ . Hence, we obtain a quasi-hereditary structure on this component, by defining a partial order as follows (see [5] or [6] for more details on quasi-hereditary algebras). Let  $v_j$  be the vertex of  $Q_v$  corresponding to the simple module  $S_j$ . Then we define  $S_j < S_i$ , for  $i \neq j$ , if there is a path from  $v_j$  to  $v_i$ , where  $S_i$  is the simple  $A_0$ -module corresponding to the vertex  $v_i$ . The standard modules with respect to this order are the projective indecomposable modules and the costandard modules are the simple modules. Therefore,  $(A_0, \leq)$  is a quasi-hereditary algebra as a product of quasi-hereditary algebras.

The Cartan matrix of a component  $A_v$  of  $A_0$  is a lower triangular matrix with diagonal elements equal to 1. Since  $A_0$  is the product of its components, we have that the following standard result for quasi-hereditary algebras holds for  $A_0$  (cf. [10]).

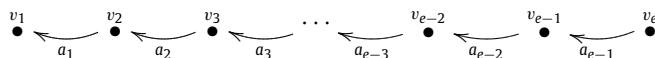
**Proposition 5.12.** *Let  $\Gamma$  be a Brauer tree of type  $(m, e)$  and let  $A_\Gamma$  be a graded basic Brauer tree algebra whose tree is  $\Gamma$  and whose grading is constructed by taking Green's walk around  $\Gamma$ . If  $A_0$  is the subalgebra of  $A_\Gamma$  consisting of elements in degree 0, then the Cartan matrix of  $A_0$  is a lower triangular matrix with diagonal elements equal to 1 and with determinant equal to 1.*

Quasi-hereditary algebras have finite global dimension (cf. [6]), hence,  $A_0$  has a finite global dimension. We give an upper bound for the global dimension of the quasi-hereditary algebra  $A_0$ . Let  $A_v$  be a connected component of  $A_0$  and let  $Q_v$  be the quiver of  $A_v$ . Let  $l(Q_v)$  be the length of the rooted tree  $Q_v$ , i.e. the index of the last level of  $Q_v$ . The global dimension of  $A_v$  is at most  $l(Q_v)$ . This can be easily proved by looking at the projective dimensions of the simple  $A_v$ -modules. One starts at the bottom of the tree and works by induction on the distance of a vertex from the bottom of the tree and uses the formula  $\text{p.d. } S = 1 + \text{p.d. rad } P_S$ , where  $S$  is a simple module and  $P_S$  its projective cover.

**Proposition 5.13.** *Let  $A_\Gamma$ ,  $A_0$  and  $Q_v$  be as above. Then,*

$$\text{gl.dim. } A_0 \leq \max\{l(Q_v) \mid Q_v \text{ a component of the quiver of } A_0\}.$$

Note that the upper bound is achieved if the relations of  $A_0$  are maximal possible in the sense that the product of every two arrows is equal to zero. This happens in the case of a Brauer line where the subalgebra  $A_0$  is given by the quiver



and the relations are  $a_i a_{i-1} = 0$ ,  $i = 2, 3, \dots, e - 1$ . The other extreme is the case when there are no relations, that is when  $A_0$  is hereditary. Then  $A_0$  has global dimension  $\leq 1$ , on the other hand  $l(Q_v)$  can be arbitrarily large.

## 6. The graded Cartan matrix

Let  $A_\Gamma$  be a basic Brauer tree algebra of type  $(m, e)$  given by the quiver  $Q$  and relations  $I$ . We have seen that  $A_\Gamma$  is a graded algebra. Let  $S_1, S_2, \dots, S_e$  be the simple  $A_\Gamma$ -modules corresponding to the vertices of the quiver  $Q$ . We assume that the simple modules are enumerated by taking Green's walk around  $\Gamma$ . We define the graded Cartan matrix  $C$  of  $A_\Gamma$  to be the  $(e \times e)$ -matrix with entries from the ring  $\mathbb{Z}[q, q^{-1}]$  given by

$$c_{ij} = C(S_i, S_j) := \sum_{l \in \mathbb{Z}} q^l \dim \text{Hom}_{A_\Gamma\text{-gr}}(P_{S_i}, P_{S_j}(l)),$$

where  $P_{S_i}$  is the projective cover of  $S_i$ .

Note that the coefficient of  $q^l$  is equal to the number of times that  $S_i$  appears in degree  $l$  as a composition factor of  $P_{S_j}$ .

**Proposition 6.1.** *Let  $A_\Gamma$  be a graded basic Brauer tree algebra whose tree is  $\Gamma$  and with grading constructed by taking Green's walk around  $\Gamma$ . Let  $S_i$  and  $S_j$  be simple modules corresponding to vertices  $v_i$  and  $v_j$  of the quiver  $Q$  of  $A_\Gamma$ . Then*

- (i) if  $S_i$  and  $S_j$  do not belong to the same cycle of  $Q$ , then  $c_{ij} = 0$ ;
- (ii) if  $S_i$  belongs to the exceptional cycle, then

$$c_{ii} = 1 + q^e + q^{2e} + \dots + q^{me},$$

if  $S_i$  does not belong to the exceptional cycle, then

$$c_{ii} = 1 + q^{me}.$$

(iii) if  $i \neq j$  and  $S_i$  and  $S_j$  belong to the same non-exceptional cycle, then

$$\begin{aligned} i > j &\Rightarrow c_{ij} = 1, \\ i < j &\Rightarrow c_{ij} = q^{me}; \end{aligned}$$

(iv) if  $i \neq j$  and  $S_i$  and  $S_j$  belong to the exceptional cycle, then

$$\begin{aligned} i > j &\Rightarrow c_{ij} = q^{i-j} + q^{i-j+e} + \dots + q^{i-j+(m-1)e}, \\ i < j &\Rightarrow c_{ij} = q^{e-(j-i)} + q^{2e-(j-i)} + \dots + q^{me-(j-i)}. \end{aligned}$$

**Proof.** Since the projective cover of  $S_j$  is spanned by the paths ending at  $S_j$ , we conclude that the exponents of the non-zero terms of  $c_{ij}$  are exactly the degrees of the non-zero paths starting at  $S_i$  and ending at  $S_j$ . Case (i) is obvious, because  $P_{S_j}$  does not contain  $S_i$  as a composition factor. In case (ii), by Lemma 4.5 the degrees of the paths starting and ending at  $S_i$  are  $0, e, 2e, \dots, me$  when  $S_i$  belongs to the exceptional cycle, and are  $0, me$  otherwise by Lemma 4.7. In case (iii), if  $i > j$ , by Lemma 4.7 the only non-zero path from  $S_i$  to  $S_j$  has degree  $0$ . Similarly, if  $i < j$ , by Lemma 4.7 the only non-zero path from  $S_i$  to  $S_j$  has degree  $me$ . In case (iv), again by Lemma 4.5, if  $i > j$ , then the degrees of the paths from  $S_i$  to  $S_j$  are  $i-j, e+(i-j), \dots, (m-1)e+i-j$ , and if  $i < j$ , they are  $e-(j-i), 2e-(j-i), \dots, me-(j-i)$ . ■

## 7. The graded Cartan determinant

**Proposition 7.1.** Let  $A_\Gamma$  be a graded basic Brauer tree algebra whose tree  $\Gamma$  is of type  $(m, e)$  and whose grading is constructed by taking Green's walk around  $\Gamma$ . If  $C_{A_\Gamma}$  is the graded Cartan matrix of  $A_\Gamma$ , then

$$\det C_{A_\Gamma} = 1 + q^e + q^{2e} + \dots + q^{me^2}.$$

**Proof.** By [15], Proposition 5.18, the constant term of  $\det C_A$  is equal to the determinant of the Cartan matrix of  $A_0$ . We have seen that the determinant of the Cartan matrix of  $A_0$  is  $1$ . By Lemma 6.5 in [15], we also have that if  $A$  and  $B$  are two graded Brauer tree algebras of the same type  $(m, e)$ , with gradings constructed by taking Green's walk, then  $\det C_A$  is equal to  $\pm q^l \det C_B$  for some integer  $l$ . Since the constant term is equal to  $1$ , we conclude that  $l = 0$ , and that  $\det C_A = \det C_B$ . Therefore, it is enough to compute  $\det C_B$  where  $B$  is the graded basic Brauer tree algebra whose tree is the Brauer line of type  $(m, e)$  with the exceptional vertex at one of the ends (see Example 4.4(b)). We will use the same labelling of the simple  $B$ -modules as in Example 4.4(b).

If  $|i-j| > 1$ , then  $c_{ij} = 0$ , because the corresponding vertices belong to different cycles. Also,  $c_{11} = 1 + q^e + q^{2e} + \dots + q^{me}$ , and if  $i > 1$ , then  $c_{ii} = 1 + q^{me}$ . Other entries are given by  $c_{i,i+1} = q^{me}$  and  $c_{i+1,i} = 1$ . We are left to compute the following  $e \times e$  determinant

$$\det C_B = \begin{vmatrix} \alpha & \beta & & & \\ 1 & \gamma & \beta & & \\ & 1 & \gamma & \beta & \\ & & & \ddots & \\ & & & 1 & \gamma & \beta \\ & & & & 1 & \gamma \end{vmatrix}$$

where  $\alpha = 1 + q^e + q^{2e} + \dots + q^{me}$ ,  $\beta = q^{me}$ ,  $\gamma = 1 + q^{me}$  and the omitted entries are all equal to zero. If  $d_l$  is the determinant of the  $l \times l$  block in the lower right corner, then from  $d_0 = 1$ ,  $d_1 = \gamma$  and the recursion

$$d_l = \gamma d_{l-1} - \beta d_{l-2},$$

it is easy to show that

$$d_l = 1 + q^{me} + \dots + q^{lme}.$$

Expanding the determinant along the first column gives us the desired formula

$$\det C_B = \alpha d_{e-1} - \beta d_{e-2} = 1 + q^e + q^{2e} + \dots + q^{me^2}. \quad \blacksquare$$

## 8. Brauer lines as trivial extension algebras

Let  $\Gamma$  be the Brauer line with  $e$  edges and multiplicity of the exceptional vertex equal to  $1$ . Let  $A_\Gamma$  be a basic Brauer tree algebra whose tree is  $\Gamma$  and let us assume that this algebra is graded by taking Green's walk around  $\Gamma$ . We have seen in Example 4.4(b) that with respect to such grading the graded quiver of  $A_\Gamma$  is given by



and we have that the only non-zero degree appearing in this grading is  $e$ . Therefore, we can divide every degree by  $e$  and we will still have a graded algebra whose graded quiver is

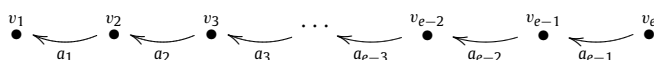


with arrows only in degrees 0 and 1. We call this procedure of dividing each degree by the same integer rescaling.

This algebra has an interesting connection with trivial extension algebras. Let  $B$  be a finite dimensional algebra over a field  $k$ . The trivial extension algebra of  $B$ , denoted  $T(B)$ , is the vector space  $B \oplus B^*$  with multiplication defined by

$$(x, f)(y, g) := (xy, xg + fy)$$

where  $x, y \in B$  and  $f, g \in B^*$  and  $B^*$  is the  $B$ -bimodule  $\text{Hom}_k(B, k)$ . This algebra is always symmetric and the map  $B \rightarrow T(B)$ , given by  $b \mapsto (b, 0)$ , is an embedding of algebras. The algebra  $T(B)$  is naturally graded by putting  $B$  in degree 0 and  $B^*$  in degree 1. This raises the question of whether the graded Brauer tree algebra  $A_\Gamma$  (with degrees rescaled by dividing by  $e$ ) is the trivial extension algebra of some algebra  $B$ ? The obvious candidate would be its subalgebra  $A_0$ . The quiver of  $A_0$  is given by



and the following proposition says that the trivial extension algebra of  $A_0$  is  $A_\Gamma$ .

**Proposition 8.1.** *Let  $A_\Gamma$  and  $A_0$  be as above. Then*

$$T(A_0) = A_0 \oplus A_0^* \cong A_\Gamma.$$

**Proof.** Let  $\{v_1^*, \dots, v_e^*, a_1^*, \dots, a_{e-1}^*\}$  be the basis of  $A_0^*$  dual to the basis  $\{v_1, \dots, v_e, a_1, \dots, a_{e-1}\}$  of  $A_0$  and let  $b_i$ ,  $i = 1, 2, \dots, e-1$ , be the arrow of the quiver of  $A_\Gamma$  starting at the vertex  $v_i$  and ending at the vertex  $v_{i+1}$ . Each  $b_i$  is in degree 1. It is now easily verified that the map given by  $a \mapsto (a, 0)$  for  $a \in A_0$  and  $b_i \mapsto (0, a_i^*)$ ,  $i = 1, 2, \dots, e-1$ , is an algebra isomorphism between  $A_\Gamma$  and  $T(A_0)$ . ■

When  $m > 1$ , we have a loop at one of the ends of the quiver of a Brauer line. This loop is in degree  $e$  and other arrows are in degrees 0 and  $me$  (see Example 4.4), hence, we have arrows in three different degrees. It follows that when  $m > 1$  these algebras cannot be seen as trivial extension algebras of their degree zero subalgebra  $A_0$  (unless  $m = 2$  and  $e = 1$ ).

## 9. Shifts of gradings

Let  $\Gamma$  be a Brauer tree of type  $(m, e)$  and let  $A_\Gamma$  be a basic Brauer tree algebra whose tree is  $\Gamma$ . We have seen that we can grade this algebra by computing the endomorphism ring of the graded complex  $T' = \bigoplus_{i=1}^e T'_i$  which we constructed by taking Green's walk around  $\Gamma$ . In other words, we got a structure of a graded algebra  $A'_\Gamma$  on  $A_\Gamma$ . Recall that this was a non-negative grading. Let  $\tilde{T}$  be the shifted graded complex  $\bigoplus_{i=1}^e T'_i \langle n_i \rangle$ , where  $n_i \in \mathbb{Z}$ ,  $i = 1, 2, \dots, e$ . The endomorphism ring of the graded complex  $\tilde{T}$  is the graded algebra  $\tilde{A}_\Gamma$  which is graded Morita equivalent to  $A'_\Gamma$  (see Definition 11.1). The question is if we can choose non-zero integers  $n_i$  in such way that the resulting grading is positive, i.e. to get such a grading in which all arrows are in positive degrees.

Let  $S_i$  and  $S_j$  be vertices of the quiver of the algebra  $\tilde{A}_\Gamma$  which belong to the same cycle, and which correspond to the summands  $T'_i \langle n_i \rangle$  and  $T'_j \langle n_j \rangle$  of  $\tilde{T}$ . We want to compute the degree of the arrow  $\alpha$  from  $S_i$  to  $S_j$ . Let  $d$  be the degree of this arrow for the graded algebra  $A'_\Gamma$ .

**Proposition 9.1.** *Let  $\alpha$  be the arrow connecting vertices  $S_i$  and  $S_j$  of the graded quiver of the graded algebra  $\tilde{A}_\Gamma$ . Then,*

$$\deg(\alpha) = d + n_i - n_j,$$

where  $d$  is the degree of the same arrow of the quiver of the graded algebra  $A'_\Gamma$ , and  $n_i$  and  $n_j$  are the shifts of  $T'_i$  and  $T'_j$  respectively.

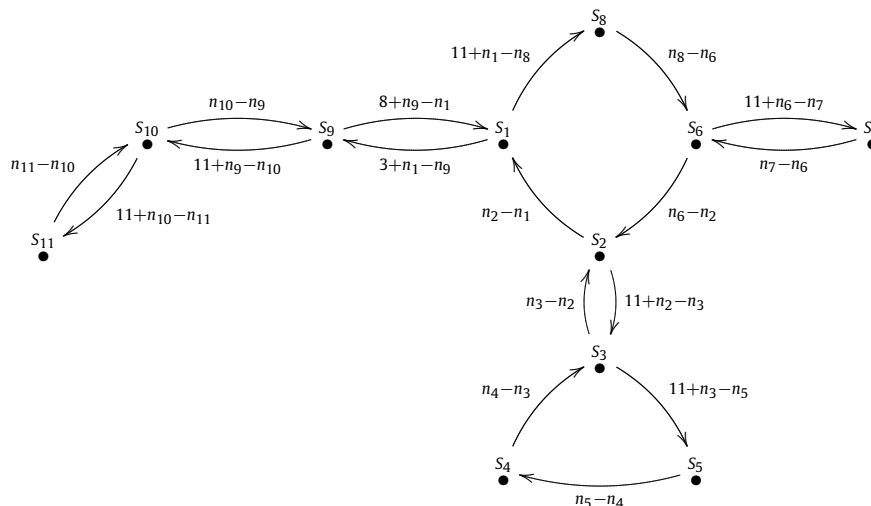
**Proof.** Let  $l \in \{i, j\}$ . Since  $T'_l$  is a complex of uniserial modules, the top of the leftmost non-zero term of  $T'_l \langle n_l \rangle$  is in degree  $d_l - n_l$ , where  $d_l$  is the degree of the top of the leftmost non-zero term of  $T'_l$ , i.e. the degree before the shift. The degree of  $\alpha$  after the shift is  $(d - n_j) - (-n_i)$ . ■

Note that when we compare graded quivers of  $A'_\Gamma$  and  $\tilde{A}_\Gamma$ , the difference is that we added  $n_i - n_j$  to the degree of the arrow from  $S_i$  to  $S_j$ . If this arrow is in degree 0, then its degree after the shift is  $n_i - n_j$ , where  $i > j$  by Lemma 4.8. The source and the target of such an arrow belong to two consecutive levels of a rooted tree. The arrow of a non-exceptional cycle whose degree was  $me$  is now in degree  $me + n_i - n_j$  where  $i < j$  by Lemma 4.7. Also, the degree of an arrow between two vertices  $S_i$  and  $S_j$  of the exceptional cycle now is  $i - j + n_i - n_j$ , if  $i > j$ , and if  $i < j$ , it is  $e - (j - i) + n_i - n_j$ .

**Proposition 9.2.** Let  $\Gamma$  be a Brauer tree of type  $(m, e)$  and let  $\tilde{T} := \bigoplus_{i=1}^e T'_i \langle n_i \rangle$  be the shifted tilting complex constructed by taking Green's walk around  $\Gamma$ . Let  $\tilde{A}_\Gamma := \text{End}_{\text{gr}_{K^b(P_{A_S})}}(\tilde{T})^{op}$ . There are positive integers  $n_i$ ,  $i = 1, 2, \dots, e$ , such that the graded algebra  $\tilde{A}_\Gamma$  is non-negatively graded with  $\deg(a) > 0$  for all homogeneous elements  $a \in \text{rad } \tilde{A}_\Gamma$ .

**Proof.** Let  $Q_v$  be an arbitrary component of the quiver of  $A_0$ , the subalgebra of  $A'_\Gamma$  of degree 0 elements, and let  $S_i$  be a vertex of  $Q_v$ . If we set  $n_i := 1 + l_i$ , where  $l_i$  is the level of the rooted tree  $Q_v$  to whom  $S_i$  belongs, then all arrows of the graded quiver  $Q_v$  are in degree 1 after the shift. The arrows of  $Q$ , the quiver of  $A_\Gamma$ , that connect two vertices of  $Q_v$  and which were not in degree 0 are still in positive degrees after the shift because  $me + n_i - n_j > 0$ . The arrows of the exceptional cycle are in the same degrees as they used to be because we set  $n_i := 1$  for each root  $S_i$ . ■

**Example 9.3.** Let  $\Gamma$  be the tree from Example 5.8. With the notation from the previous proposition the graded quiver of the basic Brauer tree algebra  $\tilde{A}_\Gamma = \text{End}_{\text{gr}_{K^b(P_{A_S})}}(\bigoplus_{i=1}^{11} T'_i \langle n_i \rangle)^{op}$  is given by



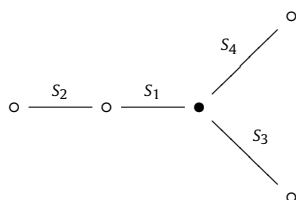
If we set  $n_1 = n_9 = 1$ ,  $n_2 = n_{10} = 2$ ,  $n_3 = n_6 = n_{11} = 3$ ,  $n_4 = n_7 = n_8 = 4$  and  $n_5 = 5$ , then all arrows are in positive degrees.

Note that the change of shifts on the summands  $T'_i$  of the tilting complex  $T'$  is the same as the change of shifts on the projective indecomposable modules of  $A'_\Gamma \cong \text{End}_{\text{gr}_{K^b(P_{A_S})}}(\bigoplus_{i=1}^e T'_i)^{op}$ . Let  $\tilde{A}_\Gamma \cong \text{End}_{\text{gr}_{K^b(P_{A_S})}}(\bigoplus_{i=1}^e T'_i \langle n_i \rangle)^{op}$ . When we change the shifts, in general, we get a different grading on  $A'_\Gamma$  and the resulting graded algebra  $\tilde{A}_\Gamma$  is not isomorphic to  $A'_\Gamma$  as a graded algebra. But these two graded algebras are graded Morita equivalent, i.e. there is an equivalence  $A'_\Gamma\text{-grmod} \cong \tilde{A}_\Gamma\text{-grmod}$  as we shall see in Section 11.

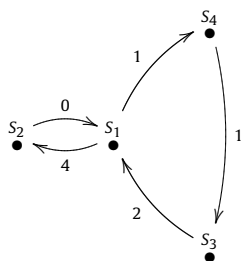
## 10. Change of the exceptional vertex

Let  $A_\Gamma$  be a basic Brauer tree algebra whose tree  $\Gamma$  has  $e$  edges and the multiplicity of the exceptional vertex equal to 1. When the multiplicity is 1, then the exceptional vertex is not uniquely defined. If we change the exceptional vertex, the algebra  $A_\Gamma$  does not change. But when constructing the tilting complex that tilts from  $A_S$  to  $A_\Gamma$  by taking Green's walk around  $\Gamma$  we start from a different vertex, and in general, the resulting tilting complex is different. Therefore, we get different gradings on  $A_\Gamma$ .

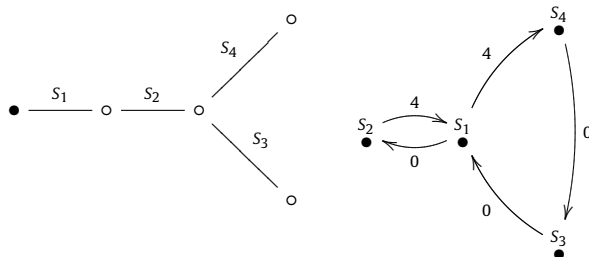
**Example 10.1.** Let  $\Gamma$  be the following Brauer tree with multiplicity 1 and let  $A_\Gamma$  be the corresponding Brauer tree algebra.



If  $T = \bigoplus_{i=1}^4 T_i$  is the tilting complex constructed by taking Green's walk around  $\Gamma$ , then the resulting graded quiver of  $A'_\Gamma$  is given by

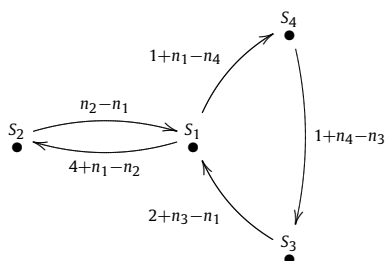


If we change the exceptional vertex, say we have a Brauer tree  $\Delta$



then the basic Brauer tree algebra  $A_\Delta$  whose tree is  $\Delta$ , is the same as  $A_\Gamma$ . The tilting complex  $D$  constructed by taking Green's walk around  $\Delta$  is different from  $T$  and it gives us a new grading on  $A_\Delta = A_\Gamma \cong \text{End}_{K^b(P_{A_\Delta})}(D)^{op}$ . The resulting graded quiver of the graded algebra  $A'_\Delta$  is given above.

If  $\tilde{T} := \bigoplus_{i=1}^4 T'_i(n_i)$  is a graded complex given by shifting the summands of  $T'$ , then from [Proposition 9.1](#) we get a grading on  $A_\Gamma \cong \text{End}_{K^b(P_{A_\Delta})}(T)^{op}$  and the resulting graded quiver of the graded algebra  $\tilde{A}_\Gamma$  is given by



If we set  $n_1 = 3$ ,  $n_2 = 7$ ,  $n_3 = 1$  and  $n_4 = 0$ , we see that the resulting grading on  $A_\Gamma$  is the same as the grading that we got by taking Green's walk around  $\Delta$ .

In the previous example we had two different gradings on  $A_\Gamma$ , but we were able, by changing shifts of the summands of the graded tilting complex  $T'$ , to move from one grading to another via graded Morita equivalence ([Definition 11.1](#)). We will prove in the next section that this holds for all Brauer tree algebras, regardless of the multiplicity of the exceptional vertex.

## 11. Classification of gradings

In this section we classify, up to graded Morita equivalence and rescaling, all gradings on an arbitrary Brauer tree algebra with  $n$  edges (we change the notation from  $e$  to  $n$  so we can use letter  $e$  for idempotents) and the multiplicity of the exceptional vertex equal to  $m$ .

For a finite dimensional  $k$ -algebra  $A$ , there is a correspondence between gradings on  $A$  and homomorphisms of algebraic groups from  $\mathbf{G}_m$  to  $\text{Aut}(A)$ , where  $\mathbf{G}_m$  is the multiplicative group  $k^*$  of the field  $k$ . For each grading  $A = \bigoplus_{i \in \mathbb{Z}} A_i$  there is a homomorphism of algebraic groups  $\pi : \mathbf{G}_m \rightarrow \text{Aut}(A)$  where an element  $x \in k^*$  acts on  $A_i$  by multiplication by  $x^i$  (see [\[15\]](#), Section 5). If  $A$  is graded and  $\pi$  is the corresponding homomorphism, we will write  $(A, \pi)$  to denote that  $A$  is graded with grading  $\pi$ .



**Definition 11.1.** Let  $(A, \pi)$  and  $(A, \pi')$  be two gradings on a finite dimensional  $k$ -algebra  $A$ , and let  $S_1, S_2, \dots, S_r$  be the isomorphism classes of simple  $A$ -modules. We say that  $(A, \pi)$  and  $(A, \pi')$  are graded Morita equivalent if there exist integers  $d_{ij}$ , where  $1 \leq j \leq \dim S_i$  and  $1 \leq i \leq r$ , such that the graded algebras  $(A, \pi')$  and  $\text{Endgr}_{(A, \pi)}(\bigoplus_{i,j} P_i \langle d_{ij} \rangle)^{\text{op}}$  are isomorphic, where  $P_i$  denotes the projective cover of  $S_i$ .

Note that two graded algebras are graded Morita equivalent if and only if their categories of graded modules are equivalent.

We now give some background on algebraic groups (more details can be found in [4]). An algebraic torus is a linear algebraic group isomorphic to  $\mathbf{G}_m^n = \mathbf{G}_m \times \dots \times \mathbf{G}_m$  ( $n$  factors) for some  $n \geq 1$ . A maximal torus in an algebraic group  $G$  is a closed subgroup of  $G$  which is a torus but is not contained in any larger torus. Tori are contained in  $G^0$ , the connected component of  $G$  that contains the identity element. For a given torus  $T$ , a cocharacter of  $T$  is a homomorphism of algebraic groups from  $\mathbf{G}_m$  to  $T$ . A cocharacter of an algebraic group  $G$  is a homomorphism of algebraic groups from  $\mathbf{G}_m$  to  $T$ , where  $T$  is a maximal torus of  $G$ . We say that cocharacters  $\pi$  and  $\pi'$  of  $G$  are conjugate if there exists  $g \in G$  such that  $\pi'(x) = g\pi(x)g^{-1}$  for all  $x \in \mathbf{G}_m$ . We see that a grading on a finite dimensional algebra  $A$  can be seen as a cocharacter  $\pi : \mathbf{G}_m \rightarrow \text{Aut}(A)$ . We will use the same letter  $\pi$  to denote the corresponding cocharacter of  $\text{Out}(A)$ , which is given by composition of  $\pi$  and the canonical surjection.

The following proposition tells us how to classify all gradings on  $A$  up to graded Morita equivalence.

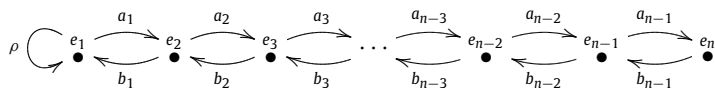
**Proposition 11.2** ([15], Corollary 5.9). *Two graded algebras  $(A, \pi)$  and  $(A, \pi')$  are graded Morita equivalent if and only if the corresponding cocharacters  $\pi : \mathbf{G}_m \rightarrow \text{Out}(A)$  and  $\pi' : \mathbf{G}_m \rightarrow \text{Out}(A)$  are conjugate.*

From this proposition we see that in order to classify gradings on  $A$  up to graded Morita equivalence, we need to compute maximal tori in  $\text{Out}(A)$ . Let  $\text{Out}^K(A)$  be the subgroup of  $\text{Out}(A)$  of those automorphisms fixing the isomorphism classes of simple  $A$ -modules. Since  $\text{Out}^K(A)$  contains  $\text{Out}^0(A)$ , the connected component of  $\text{Out}(A)$  that contains the identity element, we have that maximal tori in  $\text{Out}(A)$  are actually contained in  $\text{Out}^K(A)$ .

We start by computing  $\text{Out}^K(A_\Gamma)$  for a Brauer tree algebra  $A_\Gamma$  of type  $(m, n)$ , where  $m > 1$ . We remark here that for the case  $m = 1$ ,  $\text{Out}^K(A_\Gamma)$  has been computed in Lemma 4.3 in [16]. Although the proof of this Lemma in [16] seems to be incomplete, the result is correct if one assumes that the ground field  $k$  is an algebraically closed field. The same result, namely that  $\text{Out}^K(A_\Gamma) \cong k^*$  when  $m = 1$ , follows directly from our computation below for  $m > 1$ , if we disregard the loop  $\rho$  which does not appear if  $m = 1$ .

Let  $A_\Gamma$  be a basic Brauer tree algebra whose tree  $\Gamma$  is of type  $(m, n)$  where  $m > 1$ . Since  $\text{Out}^K(A_\Gamma)$  is invariant under derived equivalence (cf. [12], Section 4), in order to compute this group for any Brauer tree algebra  $A_\Gamma$  whose tree is of type  $(m, n)$ , it is sufficient to compute this group for the Brauer line of the same type.

Let  $A_\Gamma$  be a basic Brauer tree algebra of type  $(m, n)$  whose tree is the Brauer line with  $n$  edges and with the exceptional vertex at the end of the line. The quiver of  $A_\Gamma$  is given by



and relations are given by  $a_i a_{i+1} = b_{i+1} b_i = 0$  ( $i = 1, 2, \dots, n-2$ ),  $\rho^m = a_1 b_1$ ,  $\rho a_1 = b_1 \rho = 0$  and  $a_i b_i = b_{i-1} a_{i-1}$  ( $i = 2, 3, \dots, n-1$ ).

In order to simplify calculation let us set  $t_i := a_i + b_i$ ,  $i = 1, 2, \dots, n-1$ . Then,  $a_i = e_i t_i$  and  $b_i = e_{i+1} t_i$ , for  $i = 1, 2, \dots, n-1$ . Then the relations become  $t_i e_{i+1} t_{i+1} = t_{i+1} e_{i+1} t_i = 0$  ( $i = 1, 2, \dots, n-2$ ),  $t_i^3 = 0$  ( $i = 1, 2, \dots, n-1$ ),  $\rho e_1 t_1 = e_2 t_1 \rho = 0$ ,  $e_1 t_1^2 = \rho^m$ ,  $e_i t_i^2 = t_{i-1}^2 e_i$  ( $i = 2, 3, \dots, n-1$ ). Also, we have that  $e_i t_i = t_i e_{i+1}$  and  $e_{i+1} t_i = t_i e_i$ ,  $i = 1, 2, \dots, n-1$ . Let us write  $t_0^2 := \rho^m$ .

Let  $\varphi$  be an arbitrary automorphism of  $A_\Gamma$  that fixes isomorphism classes of simple  $A_\Gamma$ -modules. We will compose this automorphism with suitably chosen inner automorphisms in order to get an automorphism  $\phi$ , which represents the same element as  $\varphi$  in the group of outer automorphisms, but which is suitable for our computation.

If  $\{e_1, e_2, \dots, e_n\}$  is a complete set of orthogonal primitive idempotents, then  $\{\varphi(e_1), \varphi(e_2), \dots, \varphi(e_n)\}$  is also a complete set of orthogonal primitive idempotents. From classical ring theory (cf. [9], Theorem 3.10.2) we know that these two sets are conjugate. Hence, when we compose  $\varphi$  with a suitably chosen inner automorphism we get that  $x^{-1} \varphi(e_i) x = e_{\sigma(i)}$ , for all  $i$ , where  $\sigma$  is some permutation. Since  $\varphi$  fixes isomorphism classes of simple modules, we can assume that, for all  $i$ ,  $\varphi(e_i) = e_i$ .

Since  $\varphi(\text{rad } A_\Gamma) \subset \text{rad } A_\Gamma$  we have that

$$\varphi(t_i) = \sum_{j=1}^{n-1} A_{ij} e_j t_j + \sum_{j=1}^{n-1} B_{ij} e_{j+1} t_j + \sum_{j=0}^{n-1} C_{ij} e_{j+1} t_j^2 + \sum_{j=1}^{m-1} D_{ij} \rho^j,$$

where  $A_{ij}$ ,  $B_{ij}$ ,  $C_{ij}$  and  $D_{ij}$  are scalars. If  $i > 1$ , then from  $e_1 t_1 = 0$  we get that  $D_{ij} = 0$  for  $j = 1, 2, \dots, m-1$ . If  $i = 1$ , then from  $e_1 t_1 = t_1 e_2$  we get that  $D_{1j} = 0$ ,  $j = 1, 2, \dots, m-1$ . In both cases we have that  $D_{ij} = 0$  for all  $i$  and all  $j$ .

If  $l \notin \{i, i+1\}$ , then  $\varphi(e_l t_i) = 0$  and  $\varphi(t_i e_l) = 0$ . From the first equality we get that  $A_{il} = B_{i, l-1} = 0$  and from the second equality we get that  $B_{il} = A_{i, l-1} = 0$ . Subsequently, we have that  $C_{il} = 0$  for  $l \notin \{i, i-1\}$ . Therefore, we must have that

$$\varphi(t_i) = A_{ii} e_i t_i + B_{ii} e_{i+1} t_i + C_{i, i-1} e_i t_{i-1}^2 + C_{ii} e_{i+1} t_i^2.$$

From  $e_i t_i = t_i e_{i+1}$  it follows that  $\varphi(e_i)\varphi(t_i) = \varphi(t_i)\varphi(e_{i+1})$ , and we have

$$C_{i,i-1} = C_{ii} = 0.$$

Also,  $A_{ii} \neq 0$  and  $B_{ii} \neq 0$ . If one of them would be zero, then  $\varphi(t_i^2) = 0$ , which is in contradiction with  $\varphi$  being an automorphism.

Assume now that

$$\varphi(\rho) = \sum_{j=1}^{m-1} D_j \rho^j + \sum_{j=1}^{n-1} E_j e_j t_j + \sum_{j=1}^{n-1} F_j e_{j+1} t_j + \sum_{j=0}^{n-1} G_j e_{j+1} t_j^2,$$

for some scalars  $D_j, E_j, F_j$  and  $G_j$ .

For  $l > 2$ , from  $e_l \rho = 0$  we have that  $\varphi(e_l)\varphi(\rho) = 0$  and it follows that  $E_l = F_{l-1} = 0$ , for  $l = 2, 3, \dots, m-1$ . Similarly, from  $\rho e_l = 0$  we have that  $F_l = E_{l-1} = 0$ , for  $l = 2, 3, \dots, m-1$  and consequently, we have that  $G_j = 0$  for all  $j$ . Therefore, we get

$$\varphi(\rho) = \sum_{j=1}^m D_j \rho^j.$$

Note that  $D_1 \neq 0$ , because if  $D_1 = 0$ , then  $\varphi(\rho^m) = 0$ , which again contradicts the fact that  $\varphi$  is an automorphism.

We note here that the remaining relations, such as  $\rho e_1 t_1 = 0$ , impose no further restrictions to the coefficients of  $\varphi$ .

Gathering all the information we got, we conclude that, up to inner automorphism, an arbitrary automorphism that fixes isomorphism classes of simple  $A_R$ -modules has the following action on a set of generators  $\{e_i, t_i, \rho \mid i = 1, 2, \dots, n; l = 1, 2, \dots, n-1\}$

$$\varphi(e_i) = e_i, \quad \varphi(t_i) = A_{ii} e_i t_i + B_{ii} e_{i+1} t_i, \quad \varphi(\rho) = \sum_{j=1}^m D_j \rho^j.$$

To compute  $\text{Out}^K(A_R)$ , for each automorphism  $\varphi$  we need to find an automorphism which is in the same class as  $\varphi$  in  $\text{Out}^K(A_R)$ , and which acts by scalar multiplication on as many of the above generators as possible. In order to do that we will compose  $\varphi$  with suitably chosen inner automorphisms.

First of all, let us see how an arbitrary inner automorphism acts on our set of generators.

Let  $y$  be an arbitrary invertible element in  $A_R$ . Then,

$$y = \sum_{j=1}^n l_j e_j + \sum_{j=1}^{n-1} s_j e_j t_j + \sum_{j=1}^{n-1} r_j e_{j+1} t_j + \sum_{j=1}^{n-1} p_j e_{j+1} t_j^2 + \sum_{j=1}^m q_j \rho^j,$$

where  $l_i \neq 0, i = 1, \dots, n$ . From  $yy^{-1} = 1$ , we easily compute scalars  $s'_j, r'_j, p'_j, q'_j$ , in

$$y^{-1} = \sum_{j=1}^n l_j^{-1} e_j + \sum_{j=1}^{n-1} s'_j e_j t_j + \sum_{j=1}^{n-1} r'_j e_{j+1} t_j + \sum_{j=1}^{n-1} p'_j e_{j+1} t_j^2 + \sum_{j=1}^m q'_j \rho^j.$$

The inner automorphism given by  $y$  has the following action on  $\rho$

$$\begin{aligned} f_y(\rho) &:= y \rho y^{-1} = \left( l_1 \rho + \sum_{j=1}^m q_j \rho^{j+1} \right) y^{-1} \\ &= \rho + \sum_{j=1}^m l_1 q'_j \rho^{j+1} + \sum_{j=1}^m l_1^{-1} q_j \rho^{j+1} + \sum_{j=1}^m q_j \rho^{j+1} \sum_{i=1}^m q'_i \rho^i \\ &= \rho + \rho \left( \sum_{j=1}^m l_1 q'_j \rho^j + \sum_{j=1}^m l_1^{-1} q_j \rho^j + \sum_{j=1}^m q_j \rho^j \sum_{i=1}^m q'_i \rho^i + s_1 r'_1 \rho^m \right) = \rho. \end{aligned}$$

Therefore, an arbitrary inner automorphism fixes  $\rho$ . From now on, we will use inner automorphisms given by elements of the form

$$x = \sum_{j=1}^n l_j e_j.$$

They will be enough to get a class representative of  $\varphi$  in  $\text{Out}^K(A_R)$  that is easy to work with.

If  $f_x$  is the inner automorphism given by an invertible element  $x$ , then we easily get that

$$f_x(t_i) = l_i l_{i+1}^{-1} e_i t_i + l_i^{-1} l_{i+1} e_{i+1} t_i.$$

When we compose  $f_x$  and  $\varphi$  we get that

$$f_x \circ \varphi(\rho) = \sum_{j=1}^m D_j \rho^j,$$

$$f_x \circ \varphi(t_i) = A_{ii} l_i l_{i+1}^{-1} e_i t_i + B_{ii} l_i^{-1} l_{i+1} e_{i+1} t_i,$$

for all  $i$ , and

$$f_x \circ \varphi(e_i) = e_i.$$

We want to choose  $l_i$ 's so that we get

$$A_i := A_{ii} l_i l_{i+1}^{-1} = B_{ii} l_i^{-1} l_{i+1}$$

for  $i = 1, 2, \dots, n-1$ . To do this we need to choose  $l_i$ 's in such way that the following equality holds for all  $i$

$$A_{ii} B_{ii}^{-1} = l_i^{-2} l_{i+1}^2.$$

We can choose  $l_1 = 1$  and then inductively, assuming that we have chosen  $l_1, l_2, \dots, l_i$ , because we are working over an algebraically closed field, we get  $l_{i+1}$  from  $A_{ii} B_{ii}^{-1} l_i^2 = l_{i+1}^2$ .

If we choose such  $x$ , then the map  $\varphi_1 := f_x \circ \varphi$  has the following action on our generating set

$$\varphi_1(e_i) = e_i, \quad \varphi_1(\rho) = \sum_{j=1}^m D_j \rho^j, \quad \varphi_1(t_i) = A_i t_i.$$

From the relations  $e_i t_{i-1}^2 = t_i^2 e_i$ , for  $i = 2, 3, \dots, n-1$ , we get that  $A_1^2 = A_2^2 = \dots = A_{n-1}^2$ . We can assume that  $A_1 = A_2 = \dots = A_{n-1}$ , because if not, then by multiplying  $\varphi_1$  by an inner automorphism given by  $x_1 = \sum_{i=1}^n r_i e_i$ , where we set  $r_n = r_{n-1} = 1$  and then for  $i < n-1$ , we set  $r_i = -r_{i+1}$  if  $A_i = -A_{n-1}$ , and  $r_i = r_{i+1}$  if  $A_i = A_{n-1}$ , we get a new automorphism  $\varphi_2$  such that  $\varphi_2(t_i) = A_1 t_i$ , for all  $i$ . Also from the relation  $\rho^m = e_1 t_1^2$  we get that  $A_1^2 = D_1^m$ . This means that for a fixed  $D_1$  we have two choices for  $A_1$ , since there are two square roots of  $D_1^m$ . These two values of  $A_1$  will give us two different automorphisms, but as before, we can assume that after multiplying by an appropriate inner automorphism these two automorphisms represent the same automorphism in  $\text{Out}^K(A_r)$ .

We started with an arbitrary automorphism  $\varphi$  that fixes the isomorphism classes of simple  $A_r$ -modules and we showed that in the group  $\text{Out}^K(A_r)$  it represents the same class as the element  $\phi$  whose action is given by

$$\phi(e_i) = e_i, \quad \phi(\rho) = \sum_{j=1}^m D_j \rho^j, \quad \phi(t_s) = A_s t_s,$$

where  $A_i^2 = D_1^m$ ,  $i = 1, \dots, n$ , and  $s = 1, \dots, n-1$ .

Therefore, every element in  $\text{Out}^K(A_r)$  is uniquely determined by its action on  $\rho$ , i.e. it is uniquely determined by an  $m$ -tuple  $(D_1, D_2, \dots, D_m)$  where  $D_1 \neq 0$ . The map  $\theta$  that assigns to each  $m$ -tuple  $D = (D_1, D_2, \dots, D_m)$  an isomorphism  $\phi_D$ , where  $\phi_D(\rho) = \sum_{j=1}^m D_j \rho^j$ , is an isomorphism of groups. But what is the group structure on the set  $k^* \times \underbrace{k \times k \times \dots \times k}_{m-1}$ ,

where  $k^*$  denotes non-zero elements of  $k$ ? If  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$  and  $\beta = (\beta_1, \beta_2, \dots, \beta_m)$  are two  $m$ -tuples and  $\phi_\alpha$  and  $\phi_\beta$  are the corresponding automorphisms, then  $\phi_\beta \circ \phi_\alpha = \phi_{\beta * \alpha}$  gives us the definition of the group operation  $*$  on  $k^* \times \underbrace{k \times k \times \dots \times k}_{m-1}$ . Computing  $\phi_\beta \circ \phi_\alpha$  gives us that

$$\beta * \alpha := \left( \sum_{i=1}^l \alpha_i \left( \sum_{\substack{k_1 + \dots + k_l = l \\ k_1, \dots, k_l > 0}} \beta_{k_1} \beta_{k_2} \dots \beta_{k_l} \right) \right)_{l=1}^m. \quad (11.1)$$

Here are first few coordinates explicitly

$$\beta * \alpha = (\alpha_1 \beta_1, \alpha_1 \beta_2 + \alpha_2 \beta_1^2, \alpha_1 \beta_3 + 2\alpha_2 \beta_1 \beta_2 + \alpha_3 \beta_1^3, \dots).$$

**Definition 11.3.** We define  $H_m$  to be the group  $(k^* \times \underbrace{k \times k \times \dots \times k}_{m-1}, *)$ , where the multiplication  $*$  is given by the above

Eq. (11.1).

The identity element of  $H_m$  is  $(1, 0, \dots, 0)$ , and this element corresponds to the class of inner automorphisms. The inverse element of an arbitrary  $m$ -tuple is easily computed inductively from the definition of  $*$ . Associativity is verified after elementary, but tedious computation.

**Lemma 11.4.** The group  $H_m$  is isomorphic to the group of automorphisms of the polynomial algebra  $k[x]/(x^{m+1})$ .

**Proof.** An arbitrary automorphism  $f$  from  $\text{Aut}(k[x]/(x^{m+1}))$  is given by its action on  $x$ . Since it has to be surjective, and  $f(x)^{m+1} = 0$ , we have that  $x$  has to be mapped to a polynomial  $d_1x + d_2x^2 + \cdots + d_mx^m$ , where  $d_1 \neq 0$ . Therefore, every automorphism of  $\text{Aut}(k[x]/(x^{m+1}))$  is given by a unique  $m$ -tuple  $(d_1, d_2, \dots, d_m)$  where  $d_1 \neq 0$ . The structure of a group on the set of all such  $m$ -tuples is the same as for the group  $H_m$ . ■

Since the group  $\text{Out}^K(A)$  is invariant under derived equivalence and Brauer tree algebras of the same type are derived equivalent, we have the following theorem.

**Theorem 11.5.** *Let  $\Gamma$  be a Brauer tree of type  $(m, n)$  and let  $A_\Gamma$  be a basic Brauer tree algebra whose tree is  $\Gamma$ . Then*

$$\text{Out}^K(A_\Gamma) \cong \text{Aut}(k[x]/(x^{m+1})) \cong H_m.$$

We see that the group of outer automorphisms that fix isomorphism classes of simple modules of an arbitrary Brauer tree algebra depends only on the multiplicity  $m$  of the exceptional vertex and does not depend on the number of edges. If we take an arbitrary Brauer tree  $\Gamma$ , and if  $\Gamma'$  is any connected Brauer subtree of  $\Gamma$  that contains the exceptional vertex, then the corresponding Brauer tree algebras  $A_\Gamma$  and  $A_{\Gamma'}$  have isomorphic groups of outer automorphisms that fix isomorphism classes of simple modules. If we take the subtree  $\Gamma'$  to be the exceptional vertex with one edge adjacent to it, we get  $A_{\Gamma'} = k[x]/(x^{m+1})$ .

**Corollary 11.6.** *Let  $\Gamma$  be a Brauer tree of type  $(m, n)$  and let  $\Gamma'$  be an arbitrary connected Brauer subtree of  $\Gamma$  that contains the exceptional vertex. If  $A_\Gamma$  and  $A_{\Gamma'}$  are basic Brauer tree algebras whose trees are  $\Gamma$  and  $\Gamma'$  respectively, then*

$$\text{Out}^K(A_\Gamma) \cong \text{Out}^K(A_{\Gamma'}) \cong \text{Aut}(k[x]/(x^{m+1})).$$

Let  $L$  be the subgroup of  $H_m$  consisting of the elements of the form  $(1, \alpha_2, \dots, \alpha_m)$  and let  $K$  be the subgroup of  $H_m$  consisting of the elements of the form  $(\alpha_1, 0, \dots, 0)$ .

**Proposition 11.7.** *The group  $H_m$  is a semidirect product of  $L$  and  $K$ , where  $L \trianglelefteq G$  is unipotent and the subgroup  $K \cong \mathbf{G}_m$  is a maximal torus in  $H_m$ .*

We see that, regardless of the multiplicity of the exceptional vertex,  $\mathbf{G}_m$  is a maximal torus in  $\text{Out}^K(A_\Gamma)$ . It follows that the maximal tori in  $\text{Out}(A_\Gamma)$  are isomorphic to  $\mathbf{G}_m$ . From this we deduce the following theorem.

**Theorem 11.8.** *Let  $\Gamma$  be an arbitrary Brauer tree and let  $A_\Gamma$  be a basic Brauer tree algebra whose tree is  $\Gamma$ . Up to graded Morita equivalence and rescaling there is unique grading on  $A_\Gamma$ .*

**Proof.** By Proposition 11.2, cocharacters of  $\text{Out}(A_\Gamma)$  give us all gradings on  $A_\Gamma$  up to graded Morita equivalence. Maximal tori in  $\text{Out}(A_\Gamma)$  are isomorphic to  $\mathbf{G}_m$  by the previous proposition. The only homomorphisms from  $\mathbf{G}_m$  to  $\mathbf{G}_m$  are given by maps  $x \mapsto x^r$ , for  $x \in \mathbf{G}_m$  and  $r \in \mathbb{Z}$ . No two of these cocharacters are conjugate, hence, all gradings on  $A_\Gamma$  up to graded Morita equivalence are parameterized by integers. Because we can move from one of these gradings to another by rescaling, we have that there is unique grading on  $A_\Gamma$  up to rescaling (dividing each degree by the same integer) and graded Morita equivalence (shifting each summand of the tilting complex given by Green's walk). ■

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## References

- [1] J.L. Alperin, Local Representation Theory, in: Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge Univ. Press, 1986.
- [2] I. Assem, D. Simson, A. Skowronski, Elements of the Representation Theory of Associative Algebras vol. 1, in: LMS Student Texts, vol. 65, Cambridge Univ. Press, 2006.
- [3] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (2) (1996).
- [4] A. Borel, Linear Algebraic Groups, second edition, in: Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.
- [5] E. Cline, B. Parshall, L. Scott, Integral and graded quasi-hereditary algebras, J. Algebra 131 (1) (1990) 126–160.
- [6] V. Dlab, C.M. Ringel, Quasi-hereditary algebras, Illinois J. Math. 33 (1989) 280–291.
- [7] J.A. Green, Walking around the Brauer Tree, J. Austr. Math. Soc. 17 (1974) 197–213.
- [8] T. Holm, Derived categories, derived equivalences and representation theory, in: Proceedings of the Summer School on Representation theory of Algebras, Finite and Reductive Groups, Babeş-Bolyai University Cluj, 1998, pp. 33–66 <http://www.iazd.uni-hannover.de/~tholm/newcluj3.ps>.
- [9] N. Jacobson, Structure of Rings, in: Colloquium Publications, vol. 37, AMS, Providence, 1964.
- [10] S. König, C. Xi, When is a cellular algebra quasi-hereditary? Math. Ann. 315 (1999) 281–293.
- [11] S. König, A. Zimmermann, Derived Equivalences for Group Rings, in: Lecture Notes in Mathematics, vol. 1685, Springer-Verlag, Berlin, 1998.
- [12] M. Linckelmann, Stable equivalence of Morita type for self-injective algebras and  $p$ -groups, Math. Z. 223 (1996) 87–100.
- [13] C. Nastasescu, F. Van Oystaeyen, Methods of Graded Rings, in: Lecture Notes in Math., vol. 1836, Springer, 2004.
- [14] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra 61 (1989) 303–317.
- [15] R. Rouquier, Automorphismes, Graduations et Catégories Triangulées, preprint, 2009 <http://people.maths.ox.ac.uk/~rouquier/papers/autograd.pdf>.
- [16] R. Rouquier, A. Zimmermann, Picard groups for derived module categories, Proc. Lond. Math. Soc. 87 (3) (2003) 197–225.
- [17] M. Schaps, E. Zakay-Ilouz, Pointed brauer trees, J. Algebra 246 (2001) 647–672.